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Asymptotic behaviour of nonlocal p-Laplacian reaction–diffusion problems

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A B S T R A C T

In this paper, we focus on studying the existence of attractors in the phase spaces $L^2(\Omega)$ and $L^p(\Omega)$ (among others) for time-dependent p-Laplacian equations with nonlocal diffusion and nonlinearities of reaction–diffusion type. Firstly, we prove the existence of weak solutions making use of a change of variable which allows us to get rid of the nonlocal operator in the diffusion term. Thereupon, the regularising effect of the equation is shown applying an argument of a posteriori regularity, since under the assumptions made we cannot guarantee the uniqueness of weak solutions. In addition, this argument allows to ensure the existence of an absorbing family in $W_0^{1,p}(\Omega)$. This leads to the existence of the minimal pullback attractors in $L^2(\Omega)$, $L^p(\Omega)$ and some other spaces as $L^{p^*-\epsilon}(\Omega)$. Relationships between these families are also established.

1. Introduction

Nonlocal problems have become of great interest in many fields by their applications. They play a key role in Medicine [8], in the industry [16,28] and last but not least, to study the behaviour of a population with accuracy [24]. Within this framework, Chipot and Rodrigues [12] analyse the behaviour of a population of bacteria in a container considering an elliptic nonlocal diffusion equation.

In the parabolic setting, an equation with nonlocal diffusion which has caught the attention of many authors has been

$$u_t - a(l(u))\Delta u = f,$$

where $a \in C(\mathbb{R}; \mathbb{R}_+)$ is bounded from below by a positive constant, and $l \in \mathcal{L}(L^2(\Omega), \mathbb{R})$.

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Despite the similarities between the heat equation and the previous one, which looks like a simple perturbation, several difficulties arise in different contexts when nonlinear diffusion appears. For instance, the existence of a Lyapunov structure is only guaranteed under suitable assumptions (see [11] for more details) or for some specific nonlocal operators (cf. [14,13]).

Regarding the above parabolic equation and its variations, a wide range of results have been published analysing comparison results between the solution of the evolution problem and stationary solutions [10, 11], the existence of global minimizers [14], the convergence of the solution of the evolution problem to a stationary solution [15], existence of pullback attractors [3,4] or the upper semicontinuous behaviour of attractors [4], amongst others.

When the Laplace operator is replaced by the p-Laplacian, there are only two papers [13,5] in which close nonlocal problems are analysed. The p-Laplacian appears in a wide range of areas in Physics (see [34,35,25, 30] for more details). In addition, we would like to highlight that analysing a p-Laplacian problem involves additional (nontrivial) difficulties compared to the study for the Laplacian (cf. [3,4,6]), since, for example, in the result of the existence of solutions, it is necessary to rescale the time in order to use monotonicity arguments for identifying the limit.

In [13] Chipot and Savitska consider

$$\frac{\partial u}{\partial t} - \nabla \cdot a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u = f,$$

fulfilled with zero Dirichlet boundary conditions. In addition to proving the existence of solutions making use a suitable change of variable (specified below, see (12)), they establish a classification of the critical points of some energy functional. In [5], the existence of the compact global attractor in $L^2(\Omega)$ is analysed when the nonlocal operator is given by $a(l(u))$ instead of $a(\|\nabla u\|_p^p)$ as considered in [13].

In this paper, we study a much more general problem than in [5], since time-dependent terms and nonlinearities of reaction–diffusion type appear here, and the obtained results (see below) are also stronger. Namely, we consider the non-autonomous nonlocal problem for the p-Laplacian

$$\begin{cases} \dfrac{du}{dt} - a(l(u))\Delta_p u = f(u) + h(t) \quad \text{in } \Omega \times (\tau, T), \\ u = 0 \quad \text{on } \partial\Omega \times (\tau, T), \\ u(x, \tau) = u_\tau(x) \quad \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $p \geq 2$,

$$a \in C(\mathbb{R}; [m, \infty)), \tag{2}$$

where $m > 0$. Observe that we do not assume any Lipschitz condition on the function a as in [3,6]. As a consequence, in the existence result we are not able to guarantee the uniqueness of weak solutions to (1). We also suppose that $l \in (L^2(\Omega))'$, which means that

$$l(u) = l_g(u) = \int_\Omega g(x)u(x)dx \quad \text{for some } g \in L^2(\Omega).$$

Furthermore, $u_\tau \in L^2(\Omega)$, $T > \tau$ and $h \in L^{p'}_{loc}(\mathbb{R}; W^{-1,p'}(\Omega))$, where p' is the conjugate exponent of p. In addition, $f \in C(\mathbb{R})$ and there exist positive constants κ, α_1 and α_2 and $q \geq 2$, such that

$$-\kappa - \alpha_1|s|^q \leq f(s)s \leq \kappa - \alpha_2|s|^q \quad \forall s \in \mathbb{R}. \tag{3}$$