J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Characterization of polynomials as solutions of certain functional equations

J.M. Almira [a,b,*]

[a] *Departamento de Ingeniería y Tecnología de Computadores, Facultad de Informática, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain*
[b] *Departamento de Matemáticas, Universidad de Jaén, E.P.S. Linares, C/Alfonso X el Sabio, 28, 23700 Linares (Jaén), Spain*

A R T I C L E I N F O

Article history:
Received 24 July 2017
Available online xxxx
Submitted by E. Saksman

Keywords:
Fréchet's theorem
Polynomials and exponential polynomials on abelian groups
Linear functional equations
Montel's theorem
Characterization problems in probability theory
Generalized functions

A B S T R A C T

In this paper we present several characterizations of ordinary polynomials as the solution sets of certain functional equations related to the equation

$$\sum_{i=0}^{m} f_i(b_i x + c_i y) = \sum_{i=1}^{n} a_i(y) v_i(x),$$

where $x, y \in \mathbb{R}^d$ and $b_i, c_i \in \mathbf{GL}_d(\mathbb{C})$, whose solution set is, typically, formed by exponential polynomials. Some of these equations are important because of their connection with the Characterization Problem of distributions in Probability Theory.

1. Introduction

The Levi-Civita functional equation, which has the form

$$f(x+y) = \sum_{i=1}^{n} a_i(y) v_i(x), \tag{1}$$

where f, a_k, v_k are complex valued functions defined on a semigroup $(\Gamma, +)$, can be restated by claiming that $\tau_y(f) \in W$ for all $y \in \Gamma$, where $W = \mathbf{span}\{v_k\}_{k=1}^{n}$ is a finite dimensional space of functions defined on Γ and $\tau_y(f)(x) = f(x+y)$.

* Correspondence to: Departamento de Ingeniería y Tecnología de Computadores, Facultad de Informática, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain.
E-mail addresses: jmalmira@um.es, jmalmira@ujaen.es.

https://doi.org/10.1016/j.jmaa.2017.11.036
0022-247X/© 2017 Elsevier Inc. All rights reserved.

If $\Gamma = \mathbb{R}^d$ for some $d \geq 1$, the equation (1) can be formulated also for distributions, since the translation operator τ_y can be extended, in a natural way, to the space $\mathcal{D}(\mathbb{R}^d)'$ of complex valued Schwartz distributions. Concretely, we can define

$$\tau_y(f)\{\phi\} = f\{\tau_{-y}(\phi)\}$$

for all $y \in \mathbb{R}^d$ and all test function ϕ. For these distributions we will also consider, in this paper, the dilation operator

$$\sigma_b(f)\{\phi\} = \frac{1}{|\det(b)|} f\{\sigma_{b^{-1}}(\phi)\},$$

where $b \in \mathbf{GL}_d(\mathbb{C})$ is any invertible matrix, $\phi \in \mathcal{D}(\mathbb{R}^d)$ is any test function, and $\sigma_{b^{-1}}(\phi)(x) = \phi(b^{-1}x)$ for all $x \in \mathbb{R}^d$.

If X_d denotes either the set of continuous complex valued functions $C(\mathbb{R}^d)$ or the set of complex valued Schwartz distributions $\mathcal{D}(\mathbb{R}^d)'$, and $f \in X_d$, then it is known that $\tau_y(f) \in W$ for all $y \in \mathbb{R}^d$, where $W = \mathbf{span}\{v_k\}_{k=1}^n$ is a finite dimensional subspace of X_d, if and only if f is equal, in distributional sense, to a continuous exponential polynomial (also named quasi-polynomial). Indeed, if we set $M = \tau(f) = \mathbf{span}\{f(\cdot + y) : y \in \mathbb{R}^d\}$, then $M \subseteq W$ is finite dimensional and translation invariant, so that Anselone–Korevaar's Theorem [8] implies that all its elements, including $f(x)$, are exponential polynomials. This was proved in 1913 by Levi-Civita [19] for the case of ordinary continuous functions (see also [18], [20] for other proofs) Furthermore, if $\{w_k\}_{k=1}^N$ is a basis of the translation invariant space M, then every $f \in M$ satisfies the family of equations

$$\tau_y f = \sum_{i=1}^N b_i(y) w_i \qquad (y \in \mathbb{R}^d).$$

Thus, in the context of distributions, it makes sense to say that $f \in \mathcal{D}(\mathbb{R}^d)'$ satisfies the Levi-Civita functional equation if there exist distributions

$$\{v_1, \cdots, v_m\} \subseteq \mathcal{D}(\mathbb{R}^d)'$$

and ordinary functions $a_i : \mathbb{R}^d \to \mathbb{C}$ such that, for every $y \in \mathbb{R}^d$

$$\tau_y(f) = \sum_{i=1}^m a_i(y) v_i. \tag{2}$$

Indeed, we can assume that $\mathbf{span}\{v_1, \cdots, v_m\}$ is translation invariant of dimension m. Then Anselone–Korevaar's theorem guarantees that $v_1, .., v_m$ and f are all of them continuous exponential polynomials. Furthermore, once this is known, we can also demonstrate that $a_1, ..., a_m$ are also continuous exponential polynomials. This follows from the fact that the translation operator $\tau_y(f)(x) = f(x + y)$ is continuous, which implies that $a_1, \cdots, a_m$ are continuous functions and then a symmetry argument (interchange x and y) will show that they are, indeed, continuous exponential polynomials.

Note that when we say that two distributions are equal, this equality is in distributional sense. Hence, if a distribution u is equal to a continuous function f, this means that the distribution is an ordinary function and it is equal almost everywhere, with respect to Lebesgue measure, to f. Thus, when we claim that a distribution is a continuous exponential polynomial, we just state equality almost everywhere in Lebesgue sense.

Please cite this article in press as: J.M. Almira, Characterization of polynomials as solutions of certain functional equations, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2017.11.036