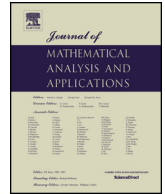




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Characterization of polynomials as solutions of certain functional equations

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ABSTRACT

In this paper we present several characterizations of ordinary polynomials as the solution sets of certain functional equations related to the equation

$$\sum_{i=0}^m f_i(b_i x + c_i y) = \sum_{i=1}^n a_i(y) v_i(x),$$

where $x, y \in \mathbb{R}^d$ and $b_i, c_i \in \mathbf{GL}_d(\mathbb{C})$, whose solution set is, typically, formed by exponential polynomials. Some of these equations are important because of their connection with the Characterization Problem of distributions in Probability Theory.

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1. Introduction

The Levi-Civita functional equation, which has the form

$$f(x + y) = \sum_{i=1}^n a_i(y) v_i(x), \quad (1)$$

where f, a_k, v_k are complex valued functions defined on a semigroup $(\Gamma, +)$, can be restated by claiming that $\tau_y(f) \in W$ for all $y \in \Gamma$, where $W = \mathbf{span}\{v_k\}_{k=1}^n$ is a finite dimensional space of functions defined on Γ and $\tau_y(f)(x) = f(x + y)$.

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If $\Gamma = \mathbb{R}^d$ for some $d \geq 1$, the equation (1) can be formulated also for distributions, since the translation operator τ_y can be extended, in a natural way, to the space $\mathcal{D}(\mathbb{R}^d)'$ of complex valued Schwartz distributions. Concretely, we can define

$$\tau_y(f)\{\phi\} = f\{\tau_{-y}(\phi)\}$$

for all $y \in \mathbb{R}^d$ and all test function ϕ . For these distributions we will also consider, in this paper, the dilation operator

$$\sigma_b(f)\{\phi\} = \frac{1}{|\det(b)|} f\{\sigma_{b^{-1}}(\phi)\},$$

where $b \in \mathbf{GL}_d(\mathbb{C})$ is any invertible matrix, $\phi \in \mathcal{D}(\mathbb{R}^d)$ is any test function, and $\sigma_{b^{-1}}(\phi)(x) = \phi(b^{-1}x)$ for all $x \in \mathbb{R}^d$.

If X_d denotes either the set of continuous complex valued functions $C(\mathbb{R}^d)$ or the set of complex valued Schwartz distributions $\mathcal{D}(\mathbb{R}^d)'$, and $f \in X_d$, then it is known that $\tau_y(f) \in W$ for all $y \in \mathbb{R}^d$, where $W = \mathbf{span}\{v_k\}_{k=1}^n$ is a finite dimensional subspace of X_d , if and only if f is equal, in distributional sense, to a continuous exponential polynomial (also named quasi-polynomial). Indeed, if we set $M = \tau(f) = \mathbf{span}\{f(\cdot + y) : y \in \mathbb{R}^d\}$, then $M \subseteq W$ is finite dimensional and translation invariant, so that Anselone–Korevaar’s Theorem [8] implies that all its elements, including $f(x)$, are exponential polynomials. This was proved in 1913 by Levi-Civita [19] for the case of ordinary continuous functions (see also [18], [20] for other proofs) Furthermore, if $\{w_k\}_{k=1}^N$ is a basis of the translation invariant space M , then every $f \in M$ satisfies the family of equations

$$\tau_y f = \sum_{i=1}^N b_i(y) w_i \quad (y \in \mathbb{R}^d).$$

Thus, in the context of distributions, it makes sense to say that $f \in \mathcal{D}(\mathbb{R}^d)'$ satisfies the Levi-Civita functional equation if there exist distributions

$$\{v_1, \dots, v_m\} \subseteq \mathcal{D}(\mathbb{R}^d)'$$

and ordinary functions $a_i : \mathbb{R}^d \rightarrow \mathbb{C}$ such that, for every $y \in \mathbb{R}^d$

$$\tau_y(f) = \sum_{i=1}^m a_i(y) v_i. \tag{2}$$

Indeed, we can assume that $\mathbf{span}\{v_1, \dots, v_m\}$ is translation invariant of dimension m . Then Anselone–Korevaar’s theorem guarantees that v_1, \dots, v_m and f are all of them continuous exponential polynomials. Furthermore, once this is known, we can also demonstrate that a_1, \dots, a_m are also continuous exponential polynomials. This follows from the fact that the translation operator $\tau_y(f)(x) = f(x + y)$ is continuous, which implies that a_1, \dots, a_m are continuous functions and then a symmetry argument (interchange x and y) will show that they are, indeed, continuous exponential polynomials.

Note that when we say that two distributions are equal, this equality is in distributional sense. Hence, if a distribution u is equal to a continuous function f , this means that the distribution is an ordinary function and it is equal almost everywhere, with respect to Lebesgue measure, to f . Thus, when we claim that a distribution is a continuous exponential polynomial, we just state equality almost everywhere in Lebesgue sense.

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