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Some variations of dual Euler–Rodrigues formula with an application to point–line geometry

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ABSTRACT

This paper examines the Euler–Rodrigues formula in dual 3-space $\mathbb{D}^3$ by analyzing its variations such as vectorial form, exponential map, point–line theory and quaternions which have some intrinsic relations. Contrary to the Euclidean case, dual rotation in dual 3-space corresponds to a screw motion in Euclidean 3-space. This paper begins by explaining dual motion in terms of the given dual axis and angle. It will then go on to express dual Euler–Rodrigues formula with algebraic methods. Furthermore, an application of dual Euler–Rodrigues formula to point–line geometry is accomplished and point–line displacement operator is obtained by dual Euler–Rodrigues formula. Finally, dual Euler–Rodrigues formula is presented with the help of dual Euler–Rodrigues parameters that is expressed as a dual quaternion.

1. Introduction

Dual space is important for the expression of dual spatial displacement which is composed of a rotation and a translation. Dual numbers and dual vectors are used in line geometry and kinematics. There are many applications concerning dual spatial displacement. For example, positions and orientations of robots has been explained by robot positional kinematics. While some of the illustrated works [1,9,10,15,20,21] use dual quaternions with geometrical approaches, the others [11,14,19] give a methodology which satisfies its application to any engineering system. In essence, the geometry of the group of rigid body motions can be explained by the elements of the matrix group $SE(3)$. The theory of $SE(3)$ and its applications robotics is given in Selig's book [17]. Elements of $SE(3)$ can be given a dual matrix. We find our motivation in this fact.

Geometrically, Euler–Rodrigues formula is a tool used to describe the motion of a rigid body. Considering this aspect, Euler–Rodrigues formula which is known as the matrix representation of spatial displacements of rigid bodies has an important role in kinematics and the mathematical description of displacements. In

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3-dimensional spaces, Euler–Rodrigues formula is used to construct a rotation matrix with the given axis and angle. The rotation matrix which is obtained when the rotation axis and rotation angle are considered as a dual vector and a dual number is called as "*Euler–Rodrigues formula in dual space*" or "*dual Euler–Rodrigues formula*".

One of the first people to define Euler–Rodrigues formula was Leonhard Euler, who found the formula as Newton–Euler equations of motion in [5]. According to the definition provided by Euler, any rigid motion leaving a point fixed may be represented by a rotation through a certain angle, around its invariant axis. With regard to Rodrigues parameters that correspond to coordinates of the rotation axis, the formula was reviewed by Olinde Rodrigues in [16]. Jian [4] has examined the Euler–Rodrigues formula, its variations and their derivations as vectors, quaternions and Lie groups and has discussed their intrinsic connections.

In point–line geometry, a point–line consists of an oriented line and an endpoint on the line. Screw systems is used to describe a point–line displacement. Bottema and Roth [3] has presented line motion in this area. In [22] Zhang and Ting have investigated geometric representation and properties of point–line displacement. They have given the expression of point–line displacement by dual quaternion algebra.

In our previous paper [7], we examine Euler–Rodrigues formula variations, split quaternionic statement and intrinsic connections in Minkowski 3-space. We obtain the Euler–Rodrigues formula in case of the rotation axis which is spacelike, timelike or lightlike.

In this paper, variations of dual Euler–Rodrigues formula, its application to point–line geometry and its representations as quaternions are examined. First, dual motion through a certain dual angle, about a dual axis is represented by using the projection of any given vector. Second, algebraic interpretations of dual Euler–Rodrigues formula are expressed with the help of exponential map from dual Lie algebra $\widehat{so}(3)$ to dual Lie group $\widehat{SO}(3)$. Later, an application of dual Euler–Rodrigues formula to point–line geometry is given and the connection between dual spatial motion and screw motion is explained. Finally, dual Euler–Rodrigues formula is computed via dual quaternion algebra.

2. Preliminaries

Dual numbers have the form $x+\varepsilon x^*$ where x, $x^* \in \mathbb{R}$ are called the *real* and *dual part* of the dual number, respectively. ε is called the *dual unit* which satisfies $\varepsilon^2 = 0$. The set of all dual numbers is denoted by $\mathbb{D}$ [8].

The set

$$\mathbb{D}^3 = \left\{\widehat{\boldsymbol{x}} = \boldsymbol{x} + \varepsilon \boldsymbol{x}^* : \boldsymbol{x}, \boldsymbol{x}^* \in \mathbb{R}^3\right\} \tag{1}$$

has a module structure on the ring $\mathbb{D}$. The elements of $\mathbb{D}^3$ is called *dual vector*. $\widehat{\boldsymbol{x}}$ is a unit dual vector if $|\widehat{\boldsymbol{x}}| = 1$. $|\widehat{\boldsymbol{x}}| = 1$ if and only if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1$ and $\langle \boldsymbol{x}, \boldsymbol{x}^*\rangle = 0$ where $\langle , \rangle$ is Euclidean inner product [8]. The symbol "$\widehat{\ }$" will use to express dual numbers and dual vectors in the following sections of the paper.

For two dual vectors $\widehat{\boldsymbol{x}} = \boldsymbol{x} + \varepsilon \boldsymbol{x}^*$ and $\widehat{\boldsymbol{y}} = \boldsymbol{y} + \varepsilon \boldsymbol{y}^*$, the scalar product $\langle , \rangle : \mathbb{D}^3 \times \mathbb{D}^3 \to \mathbb{D}$ and cross product $\wedge : \mathbb{D}^3 \times \mathbb{D}^3 \to \mathbb{D}^3$ of $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{y}}$ are defined

$$\langle \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}\rangle = \langle \boldsymbol{x}, \boldsymbol{y}\rangle + \varepsilon(\langle \boldsymbol{x}, \boldsymbol{y}^*\rangle + \langle \boldsymbol{x}^*, \boldsymbol{y}\rangle) \tag{2}$$

and

$$\widehat{\boldsymbol{x}} \wedge \widehat{\boldsymbol{y}} = \boldsymbol{x} \wedge \boldsymbol{y} + \varepsilon(\boldsymbol{x} \wedge \boldsymbol{y}^* + \boldsymbol{x}^* \wedge \boldsymbol{y}) \tag{3}$$

on dual 3-space, respectively [3].

The *Lagrange's formula* is expressed by,

$$\widehat{\boldsymbol{x}} \wedge (\widehat{\boldsymbol{y}} \wedge \widehat{\boldsymbol{z}}) = \langle \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{z}}\rangle\, \widehat{\boldsymbol{y}} - \langle \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}\rangle\, \widehat{\boldsymbol{z}} \tag{4}$$