J. Math. Anal. Appl. 459 (2018) 1040–1061

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Closed range of $\bar{\partial}$ in L^2-Sobolev spaces on unbounded domains in $\mathbb{C}^n$

Phillip S. Harrington, Andrew Raich *,1

Department of Mathematical Sciences, SCEN 309, 1 University of Arkansas, Fayetteville, AR 72701, United States

A R T I C L E   I N F O

Article history:
Received 25 April 2017
Available online 16 November 2017
Submitted by E.J. Straube

Keywords:
Unbounded domains
Weighted Sobolev spaces
Defining functions
Closed range
$\bar{\partial}$-Neumann
Weak $Z(q)$

A B S T R A C T

Let $\Omega \subset \mathbb{C}^n$ be a domain and $1 \leq q \leq n-1$ fixed. Our purpose in this article is to establish a general sufficient condition for the closed range of the Cauchy–Riemann operator $\bar{\partial}$ in appropriately weighted L^2-Sobolev spaces on $(0,q)$-forms. The domains we consider may be neither bounded nor pseudoconvex, and our condition is a generalization of the classical $Z(q)$ condition that we call weak $Z(q)$. We provide examples that explain the necessity of working in weighted spaces both for closed range in L^2 and, even more critically, in L^2-Sobolev spaces.

1. Introduction

This paper is a continuation of [12]. We suppose that $\Omega \subset \mathbb{C}^n$ is a smooth domain, and we require neither boundedness nor pseudoconvexity of Ω. Our objective to find the weakest possible sufficient condition that ensures the Cauchy–Riemann operator $\bar{\partial}$ has closed range on $(0,q)$-forms in L^2-Sobolev spaces, for a *fixed* q, $1 \leq q \leq n-1$. In [12], we proved closed range only in L^2. When Ω is bounded and pseudoconvex, our result reproduces the classical cases (e.g., Kohn [16]).

We continue to explore the weak $Z(q)$ hypothesis that we introduced in [10]. Weak $Z(q)$ (defined below) is a curvature condition on the Levi form that suffices to prove that the range of $\bar{\partial}$ is closed in $L^2_{0,q}$ or $L^2_{0,q+1}$ on bounded domains in Stein manifolds as well as unbounded domains with uniform C^3 regularity. The weak $Z(q)$ condition is a more general version than the authors' condition in [7], and is closely related to, but still more general than, related conditions in [14], [1], and [18] which have been investigated for closed

* Corresponding author.
E-mail addresses: psharrin@uark.edu (P.S. Harrington), araich@uark.edu (A. Raich).

[1] The second author was partially supported by NSF grant DMS-1405100.

https://doi.org/10.1016/j.jmaa.2017.11.017

range of $\bar{\partial}$ (or $\bar{\partial}_b$) in a variety of settings. Its name derives from the fact that it generalizes the classic $Z(q)$ condition (see [15], [4], [2], or [3]).

Unbounded domains in $\mathbb{C}^n$ may exhibit very different behavior than bounded ones. For example, Ω satisfies the classic $Z(q)$ condition when the Levi form has either at least $q+1$ negative or at least $n-q$ positive eigenvalues at every boundary point. However, on any bounded domain, there must be at least one strictly (pseudo)convex boundary point, which forces (by continuity of the eigenvalues of the Levi form) a bounded $Z(q)$ domain in $\mathbb{C}^n$ to have at least $n-q$ positive eigenvalues at every boundary point. Hence, a large class of interesting local examples (those with at least $q+1$ negative eigenvalues) cannot be realized globally as bounded domains in $\mathbb{C}^n$ (or indeed any Stein manifold). For an in depth look at the consequences of $Z(q)$ for unbounded domains, please see [11].

In order to prove closed range of $\bar{\partial}$ in L^2 on any reasonable class of unbounded domains, it is necessary to work in weighted L^2 spaces. Unlike in the bounded case, these weighted L^2 spaces are not equivalent to the unweighted spaces. A simple counterexample demonstrates the necessity of using a weight function. Suppose that Ω contains balls of arbitrarily large radii. We want to see that the closed range estimate

$$\|u\|_{L^2(\Omega)} \leq C(\|\bar{\partial}u\|_{L^2(\Omega)} + \|\bar{\partial}^*u\|_{L^2(\Omega)}) \tag{1.1}$$

cannot hold for any $C > 0$. Also $\bar{\partial}^*$ is the L^2 adjoint of $\bar{\partial}$ (see Section 2 for details on the notation). The Siegel upper space $\{(z,w) \in \mathbb{C}^{n+1} : \Im w > |z|^2\}$ satisfies the large ball condition and is the unbounded domain *par excellence* – its boundary is the Heisenberg group and it is also biholomorphic to the unit ball. By the large ball condition, there exists $z_R \in \Omega$ such that $B(z_R, R) \subset \Omega$ for every $R > 0$. Let $u_1 \in C^\infty_{0,(0,q)}(B(0,1))$ be nontrivial, and define $u_R(z) = \frac{1}{R^n} u_1\left(\frac{z-z_R}{R}\right)$. Then $u_R \in C^\infty_{0,(0,q)}(B(z_R,R)) \subset C^\infty_{0,(0,q)}(\Omega)$. If (1.1) were to hold, then

$$\|u_1\|_{L^2(\Omega)} = \|u_R\|_{L^2(\Omega)} \leq C(\|\bar{\partial}u_R\|_{L^2(\Omega)} + \|\bar{\partial}^*u_R\|_{L^2(\Omega)}) = R^{-1}C(\|\bar{\partial}u_1\|_{L^2(\Omega)} + \|\bar{\partial}^*u_1\|_{L^2(\Omega)}).$$

Since this inequality must hold for every $R > 0$, we have a contradiction. Thus, closed range estimates in L^2 are impossible on many unbounded domains, so we must consider weighted L^2 spaces. In [12], we do briefly touch upon the L^2-theory for $\bar{\partial}$ in unweighted L^2 spaces for domains that satisfy weak $Z(q)$. Gallagher and McNeal establish sufficient conditions for the closed range of $\bar{\partial}$ in L^2 unbounded, pseudoconvex domains [13].

Even if we wanted to concentrate on domains for which we can establish the unweighted L^2 theory for $\bar{\partial}$, there is no hope for any usable result in Sobolev spaces. The reason is that the Sobolev space theory is effectively useless on any interesting unbounded domain. For example, suppose that Ω contains infinitely many disjoint balls B_k of fixed radius r (as is the case in the model domain defined by $\rho(z) = \sum_{j=1}^n (\operatorname{Re} z_j)^2 - 1$ for which $\bar{\partial}$ has closed range in unweighted L^2 [12]). If we take any function $f \in C_0^\infty(B(0,r))$ and define $f_k(z) = f(z - c_k)$, where c_k is the center of B_k, then we have a sequence $\{f_k\}$ that is uniformly bounded in L^2 with no convergent subsequence. Hence, $H^1(\Omega)$ is not compact in $L^2(\Omega)$, and the Rellich Lemma fails, making any theory of Sobolev Spaces extremely problematic.

When working on weighted L^2 spaces for unbounded domains, adjoints of differential operators can introduce low order terms with unbounded coefficients. For example, if D is a differential operator and $e^{-\varphi}$ is our weight, we have

$$D^*_\varphi = e^\varphi D^* e^{-\varphi} = D^* + (\bar{D}\varphi).$$

Roughly speaking, our Sobolev spaces must be defined in such a way that multiplying by the unbounded function $\bar{D}\varphi$ is no worse that differentiating in D^*. This means that great care is required when defining Sobolev spaces. In [9], the authors developed the theory of weighted Sobolev spaces on unbounded domains building on ideas in [6] and [5]. Boundary smoothness also requires greater care, since derivatives of defining