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On the exponential large sieve inequality for sparse sequences modulo primes

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ABSTRACT

We complement the argument of M. Z. Garaev (2009) [9] with several other ideas to obtain a stronger version of the large sieve inequality with sparse exponential sequences of the form λ^{s_n}. In particular, we obtain a result which is non-trivial for monotonically increasing sequences $\mathcal{S} = \{s_n\}_{n=1}^{\infty}$ provided $s_n \leqslant n^{2+o(1)}$, whereas the original argument of M. Z. Garaev requires $s_n \leqslant n^{15/14+o(1)}$ in the same setting. We also give an application of our result to arithmetic properties of integers with almost all digits prescribed.

1. Introduction

The classical large sieve inequality, giving upper bounds on average values of various exponential and similar Dirichlet polynomials, such as

$$\sum_{q=1}^{Q} \sum_{\substack{a=1\\ \gcd(a,q)=1}}^{q} \left| \sum_{s=1}^{S} \alpha_s \exp\left(2\pi i a s/q\right) \right|^2 \quad \text{and} \quad \sum_{q=1}^{Q} \sum_{\substack{\chi \bmod q\\ \chi \text{ prim.}}} \left| \sum_{s=1}^{S} \alpha_s \chi(s) \right|^2,$$

with primitive multiplicative characters χ modulo q and arbitrary complex weights $\{\alpha_s\}_{n=1}^{S}$, has proved to be an extremely useful and versatile tool in analytic number theory and harmonic analysis, see, for example, [13,17,18].

Furthermore, if the weights α_s are supported only on elements of some sequence $\mathcal{S} = \{s_n\}_{n=1}^{T}$, which naturally occurs in many number theoretic applications, then the above sums can be written as

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$$\sum_{q=1}^{Q} \sum_{\substack{a=1\\ \gcd(a,q)=1}}^{q} \left| \sum_{n\leqslant T} \gamma_n \, \mathbf{e}_q\left(s_n\right) \right|^2 \quad \text{and} \quad \sum_{q=1}^{Q} \sum_{\substack{\chi \bmod q\\ \chi \text{ prim.}}} \left| \sum_{n\leqslant T} \gamma_n \chi(s_n) \right|^2, \tag{1.1}$$

where $\gamma_n = \alpha_{s_n}$ and

$$\mathbf{e}_q(z) = \exp(2\pi i z/q).$$

However, the power of general bounds rapidly diminishes when the sequence $\mathcal{S}$ becomes sparse.

Partially motivated by this phenomenon, and partially by applications to Mersenne numbers, Garaev and Shparlinski [10, Theorem 3.1] have introduced a modification of the large sieve, for both exponential and Dirichlet polynomials with arguments that contain exponentials from extremely sparse sequences.

In particular, in the setting of [10], the arguments of the exponentials and characters appearing in (1.1) contain exponential functions λ^{s_n} with elements of $\mathcal{S}$ rather than the elements of $\mathcal{S}$ themselves. In the case of exponential polynomials, Garaev [9] has introduced a new approach, which has led to a stronger version of the exponential large sieve inequality, improving some of the results of [10], see also [1, Lemma 2.11] and [22, Theorem 1] for several other bounds of this type. Furthermore, stronger versions of the exponential large sieve inequality for special sequences $\mathcal{S}$, such as T consecutive integers or the first T primes, can also be found in [1,10], with some applications given in [21].

Here we continue this direction and concentrate on the case of general sequences $\mathcal{S}$ without any arithmetic restriction. We introduce several new ideas which allow us to improve some results of Garaev [9]. For example, we make use of the bound of [15, Theorem 5.5] on exponential sums over small multiplicative subgroups modulo p, which hold for almost all primes p, see Lemma 3.2. We also make the method more flexible so it now applies to much sparser sequences $\mathcal{S}$ than in [9]. We believe these ideas may find more applications in similar problems.

More precisely, let us fix some integer $\lambda \geqslant 2$. For each prime number p, we let t_p denote the order of $\lambda \bmod p$. For real X and Δ we define the set

$$\mathcal{E}_\Delta(X) = \{p \leqslant X \ : \ t_p \geqslant \Delta\}.$$

Note that by a result of Erdős and Murty [8], see also (2.15), for $\Delta = X^{1/2}$, almost all primes $p \leqslant X$ belong to $\mathcal{E}_\Delta(X)$.

For integer T and two sequences of complex weights $\Gamma = \{\gamma_n\}_{n=1}^T$ and integers $\mathcal{S} = \{s_n\}_{n=1}^T$ we define the sums

$$V_\lambda(\Gamma, \mathcal{S}; T, X, \Delta) = \sum_{p\in\mathcal{E}_\Delta(X)} \max_{\gcd(a,p)=1} \left| \sum_{n\leqslant T} \gamma_n \, \mathbf{e}_p(a\lambda^{s_n}) \right|^2 .$$

These sums majorize the ones considered by Garaev [9] where each term is divided by the divisor function $\tau(p-1)$ of $p-1$. Here we obtain a new bound of the sums $V_\lambda(\Gamma, \mathcal{S}; T, X, \Delta)$ which in particular improves some bounds of Garaev [9].

The argument of Garaev [9] reduces the problem to bounding Gauss sums for which he uses the bound of Heath-Brown and Konyagin [12], that is, the admissible pair (2.1), which is defined below. In particular, for $V_\lambda(\Gamma, \mathcal{S}; T, X, X^{1/2})$ the result of Garaev [9] is nontrivial provided

$$S \leqslant X^{15/14+o(1)}. \tag{1.2}$$

Our results by-pass significantly the threshold (2.9) and allow us to replace 15/14 with any fixed $\vartheta < 2$.