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# The PML method to solve a class of potential scattering problem with tapered wave incidence ☆

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ABSTRACT

This paper concerns about the two-dimensional direct scattering problem for Schrödinger equation with a class of decaying potential under the tapered wave incidence. We focus on the numerical solution of the problem by using the perfectly matched layer (PML) method. Based on the idea of complex coordinate stretching, the boundary value problem for the PML equation is formulated. Then the variational equation to the boundary value problem is proved to exist a unique solution except possibly for a set of values of wavenumber. The numerical experiments illustrate the influence of the thickness of PML layer, the absorption parameter, the wavenumber, the control bandwidth and even the incidence angle of the tapered wave on the effectiveness of this method.



## 1. Introduction

In quantum mechanics, we often consider the scattering problem for stationary Schrödinger equation (see e.g. [4,11,13])

$$\begin{cases} (-\Delta + q(x) - k^2)u(x) = 0, \quad x \in \mathbb{R}^2, \\ u = u^i + u^s, \\ u^s = O\Big(\frac{1}{\sqrt{r}}\Big), \quad \frac{\partial u^s}{\partial r} - \mathbf{i}ku^s = o\Big(\frac{1}{\sqrt{r}}\Big), \quad r = |x| \to \infty, \end{cases} \tag{1}$$

where $q(x) > 0$ is the potential, $k \in \mathbb{R}^+$ is the wavenumber, $\mathbf{i}$ is the imaginary unit, $u^i$ is the incident wave, $u^s$ is the scattered wave, $u$ is called the total wave, the two asymptotic expressions denote the Sommerfeld boundedness condition and the Sommerfeld radiation condition satisfied by the scattered wave at infinity

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due to the physical background requirements. The direct potential scattering problem can be stated as follows: given $q(x)$, $k$ and $u^i$, determine $u$ or $u^s$ that satisfies (1).

There are two kinds of incident waves used usually in the literature: one is the plane wave (see e.g. [3,8,15]) with the form $u^i = \mathrm{e}^{\mathbf{i}kx\cdot d}$, where $d$ is the direction of propagation, the other one is the wave generated by a point source (see e.g. [9,14]), i.e. $u^i = \frac{\mathbf{i}}{4}H_0^{(1)}(k|x-y|)$, where $y$ denotes the location of the point source, $H_0^{(1)}$ is the first kind of Hankel function of order 0. However, both of the two kinds of waves tend to be idealistic from the view of application. If the actual setting of the emission device for the incident wave is considered, the tapered wave proposed in engineering may be more reasonable. Tapered wave is often chosen as the incident wave in the study of scattering from unbounded rough surface ([10,16–19]). Besides the reason of actual device mentioned above, another motivation of this choice is that if the plane incident wave is employed, most of the incident energy has to be thrown away once the truncation is made in numerical computations. In contrast, the incident energy of the tapered wave concentrates mainly in limited bandwidth, beyond which it decays exponentially. Thus most of the energy can be reserved after truncation, which is expected from the perspective of energy conversation. For the scattering problem studied in current paper, the potential is distributed in whole space and truncation is needed as well for numerical computations, which in some ways are similar to the scattering problem of rough surface. Therefore, we are inspired to employ the tapered wave to study the potential scattering problem. We adapt the Thorsos tapered wave ([16]) with the form

$$u^i(x) = \exp(\mathbf{i}k(x_1\sin\theta_i - x_2\cos\theta_i)(1+\omega(x)))\exp\Big(\frac{-(x_1+x_2\tan\theta_i)^2}{g^2}\Big), \tag{2}$$

where $x=(x_1,x_2)$, $\theta_i$ is the incident angle defined as the angle between the direction of propagation and the negative $x_2$ axis,

$$\omega(x) = \frac{1}{(kg\cos\theta_i)^2}\Big(\frac{2(x_1+x_2\tan\theta_i)^2}{g^2} - 1\Big).$$

It is easy to verify that $u^i(x)$ satisfies the nonhomogeneous Helmholtz equation

$$\Delta u^i + k^2u^i = k^2F, \quad \text{in } \mathbb{R}^2. \tag{3}$$

Here,

$$\begin{aligned} F = u^i(x)\Big\{ & -\omega^2(x) - 16\frac{(x_1\sin\theta_i - x_2\cos\theta_i)^2(x_1+x_2\tan\theta_i)^2}{k^4g^8\cos^6\theta_i} \\ & + \frac{4\mathbf{i}k(x_1\sin\theta_i - x_2\cos\theta_i)}{k^4g^4\cos^4\theta_i}\Big(1 - \frac{4(x_1+x_2\tan\theta_i)^2}{g^2}\Big)\Big\}, \end{aligned} \tag{4}$$

where $g$ is the parameter that controls the tapering and we call it “control bandwidth” in this paper. We get from (1) and (3) that the scattered wave $u^s$ obeys the equation

$$(-\Delta + q(x) - k^2)u^s = -q(x)u^i + k^2F. \tag{5}$$

Throughout this paper we assume that there exists some $r_0 > 0$ such that the potential $q(x)$ is bounded if $r < r_0$ and $q(x) = \frac{b}{r^\delta}$ if $r \geq r_0$, where $r = |x|$, $b > 0$, $\delta > 1$, both $b$ and $\delta$ are constants. It is worth pointing out that since what we study is the scattering problem in unbounded domain, the assumption on $q(x)$ for $r \geq r_0$ plays a key role in subsequent discussions. In other words, $q(x)$ for $r < r_0$ is not necessary to satisfy the decaying condition and we claim that boundedness is sufficient for it. Particularly, $q(x)$ is allowed to be