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On boundary control of the Poisson equation with the third boundary condition

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A B S T R A C T

This paper studies controllability of the Poisson equation on the unit disk in $\mathbb{C}$ subject to the third boundary condition when the control is imposed on the boundary. We use complex analytic methods to prove existence and uniqueness of the control when the parameter λ is a nonzero complex number but not a negative integer (not an eigenvalue). Otherwise, due to multiplicity of solutions to the underlying problem, when λ is a negative integer, controllability could only be obtained if proper additional conditions on the boundary are imposed.

1. Introduction

We consider the Poisson equation for functions of complex variables on the unit disk $\mathbb{D} \subset \mathbb{C}$

$$\Delta u = f \qquad \text{in} \qquad \mathbb{D}, \tag{1.1}$$

subject to the boundary condition

$$\frac{\partial}{\partial n} u + \lambda u = g \qquad \text{on} \qquad \partial\mathbb{D}, \tag{1.2}$$

where n is the outward unit normal and $\lambda \in \mathbb{C}\backslash\{0\}$. This boundary condition is known as the third boundary condition. We assume throughout that $f \in C(\overline{\mathbb{D}})$ (continuous). We investigate the optimal control problem of finding a control g in an admissible set of functions which minimizes the quadratic functional

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$$J(u,g) = \int_{\mathbb{D}} |u|^2 \, dx \, dy + \int_0^{2\pi} |g(\theta)|^2 \, d\theta, \tag{1.3}$$

for different values of the parameter $\lambda \in \mathbb{C}\backslash\{0\}$.

Most works in the literature treat the case when λ is a positive real parameter [10,15,20] known as the Robin boundary condition [18,24,23,25] for which energy methods can be used to study the problem. The case of $\lambda < 0$, also known as the Steklov problem [28,18], has been considered by [5] and explicit solutions are provided in terms of polar coordinates on a disk and it has been also studied by [3] where explicit solutions are provided in terms of spherical coordinates on a sphere. In [4], the author considered different representations of Green's functions for Laplacian boundary value problems.

The Steklov problem is an eigenvalue problem with the spectral parameter in the boundary condition and has various applications [17]. This boundary condition is relevant to the study of certain physical and biological models such as the electron energy barrier model and oxygen absorption of human lungs [18,16]. In some special cases, the Steklov spectrum can be explicitly computed as in [17] where the Steklov eigenvalues and eigenfunctions of cylinders and balls are calculated explicitly using separation of variables for a non-negative weight function $\rho \equiv 1$ on the boundary and where the eigenfunctions are given in terms of polar coordinates for the unit disk.

This paper considers however the more general case where λ is any complex valued parameter on the unit disk, and refers to this case by the "third boundary condition". Moreover the explicit solutions are provided in terms of holomorphic and anti holomorphic polynomials/functions from the perspective of complex analysis and the parameter is assumed to be any complex number rather than limiting it to be a positive or a negative real number [25]. The third boundary condition for holomorphic and harmonic functions is studied by [24,23,25] and explicit solutions are provided for the case when λ is a general complex valued function. In this paper, we utilize the explicit solutions provided by [25] for the case of a general complex valued constant λ, to obtain a boundary controllability result on the solution to the BVP.

For $\lambda > 0$, or the Robin problem on a general bounded domain, it is well known that there is a unique solution, given sufficiently regular data g. Energy methods and the Lax–Milgram theorem are usually invoked to establish the existence of a weak solution to the equation. The controllability of this problem and optimal solution in case of the Robin as well as the Dirichlet boundary conditions are also known [13,21,2] for some general domains and in the case of Steklov, there are also some studies available, among others [14,17]. These tools cannot be applied however for more general values of λ when λ is a complex number or when $\lambda(z)$ is a complex valued function. In this paper, we study controllability of the Poisson equation with the third boundary condition when the control is imposed in the boundary condition.

2. Preliminaries

2.1. Function spaces

We introduce the Sobolev space which we will use throughout the paper

$$W^{2,1}(\mathbb{D}) \equiv \{v \in L^2(\mathbb{D}) : \partial_z v \in L^2(\mathbb{D}), \partial_{\overline{z}} v \in L^2(\mathbb{D})\},$$

equipped with the norm

$$\|v\|_{W^{2,1}(\mathbb{D})} \equiv \left(\int_{\mathbb{D}} |v|^2 \, dx \, dy + \int_{\mathbb{D}} |\partial_z v|^2 \, dx \, dy + \int_{\mathbb{D}} |\partial_{\overline{z}} v|^2 \, dx \, dy \right)^{1/2},$$