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Normalized solutions for a coupled Schrödinger system with saturable nonlinearities

Xiaofei Cao [a,*], Junxiang Xu [b], Jun Wang [c], Fubao Zhang [b]

[a] *Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huaian, 223003, PR China*
[b] *Department of Mathematics, Southeast University, Nanjing, 210096, PR China*
[c] *Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu, 212013, PR China*

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ABSTRACT

We study the existence of prescribed L^2-norm solutions for the coupled Schrödinger system with saturable nonlinearities, which appears in models for propagation of a beam with two mutually incoherent components in a bulk saturable medium. Moreover, the standing waves associated to the set of minimizers have the orbital stability.

1. Introduction and main results

Nonlinear optic provides good knowledge of the transmission of light. Moreover, spatial optical solitons have recently attracted much attention, since they can maintain their shape during the propagation. As we all know, the passage of a ray along different materials induces several nonlinear effects. The propagation of a beam with two mutually incoherent components in a bulk saturable medium in the isotropic approximation can be described by the time-dependent two-component coupled nonlinear Schrödinger system with saturable nonlinearities

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi = \frac{\rho(|\Phi|^2+\Psi|^2)}{1+(|\Phi|^2+\Psi|^2)/I_0}\Phi & \text{for} \quad t>0, x\in\mathbb{R}^N, \\ -i\frac{\partial}{\partial t}\Psi = \frac{\rho(|\Phi|^2+\Psi|^2)}{1+(|\Phi|^2+\Psi|^2)/I_0}\Psi & \text{for} \quad t>0, x\in\mathbb{R}^N, \end{cases} \tag{1.1}$$

where i is the imaginary unit. Physically, Φ and Ψ denote amplitudes of the components of the beam in photorefractive crystals, ρ is the strength of the nonlinearity, I_0 is the saturation parameter and $|\Phi|^2+|\Psi|^2$

* Corresponding author.
E-mail addresses: caoxiaofei258@126.com (X. Cao), xujun@seu.edu.cn (J. Xu), wangmath2011@126.com (J. Wang), zhangfubao@seu.edu.cn (F. Zhang).

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is the total intensity created by all incoherent components of the light beam (see [7,14–17,19,20]). Obviously, finding solitary wave solutions of the form

$$\Phi(x,t) = \sqrt{\rho}e^{-i\lambda_1 t}u(x) \quad \text{and} \quad \Psi(x,t) = \sqrt{\rho}e^{-i\lambda_2 t}v(x) \tag{1.2}$$

is equivalent to solving the following coupled elliptic system

$$\begin{cases} -\Delta u + \lambda_1 u = \frac{u^2+v^2}{1+s(u^2+v^2)}u & \text{in} \quad \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \frac{u^2+v^2}{1+s(u^2+v^2)}v & \text{in} \quad \mathbb{R}^N, \\ u, v \in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{1.3}$$

where $s = \rho/I_0$. Therefore many researchers [17,18] are interested in the following general system

$$\begin{cases} -\Delta u + \lambda_1 u = \frac{\Gamma_1(\Gamma_1 u^2+\Gamma_2 v^2)}{1+s(\Gamma_1 u^2+\Gamma_2 v^2)}u & \text{in} \quad \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \frac{\Gamma_2(\Gamma_1 u^2+\Gamma_2 v^2)}{1+s(\Gamma_1 u^2+\Gamma_2 v^2)}v & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $\lambda_1, \lambda_2, \Gamma_1, \Gamma_2$ are positive constants and s is a positive parameter. It is easy to see that (1.3) is the special case of (1.4) with $\Gamma_1 = \Gamma_2 = 1$. A solution (u, v) is called nontrivial if $u \not\equiv 0$ and $v \not\equiv 0$, a solution (u, v) is semi-trivial if (u, v) is of type $(u, 0)$ or $(0, v)$. We call a nontrivial solution (u, v) positive if $u > 0$ and $v > 0$. In [17], the authors first considered the special case with $\Gamma_1 = \Gamma_2 = 1$ and $\lambda_1 = \lambda_2 := \lambda$, and proved that if $s \geq 1/\lambda$ then the system (1.4) has no nontrivial solution, and if $s \in (0, 1/\lambda)$ then the system (1.4) has a unique-up to rotations-solution, which can be given by the solution of the following scalar equation

$$-\Delta u + \lambda u = \frac{u^3}{1+su^2} \quad \text{in} \quad \mathbb{R}^N.$$

For the case with $\Gamma_1 > \Gamma_2$ and $\lambda_1 > \lambda_2$, they proved that if $s > \max\{\Gamma_1/\lambda_1, \Gamma_2/\lambda_2\}$, the system (1.4) has no nontrivial solutions and if

$$s \in \left((\Gamma_1 - \Gamma_2) / (\lambda_1 - \lambda_2), \max\{\Gamma_1/\lambda_1, \Gamma_2/\lambda_2\} \right),$$

the system (1.4) has a semi-trivial ground state solution, i.e., the least energy solution of the system (1.4) is either $(0, u)$ or $(v, 0)$. However, if $s \in (0, (\Gamma_1 - \Gamma_2)/(\lambda_1 - \lambda_2))$, they only conjectured that the system (1.4) should have semi-trivial ground state solutions. This conjecture was settled later by Mandel in [18]. Moreover, the author [18] also obtained some positive solutions and semi-nodal solutions for the system (1.4).

Recently, many authors pay attention to normalized solutions, which satisfy the constraint

$$\int_{\mathbb{R}^N} u^2\, dx = a$$

with $a > 0$ prescribed. The motivation to look for normalized solutions is that the masses are preserved along trajectories, which seems to be particularly interesting from physical point of view. This problem has first been studied for scalar equations [4,6,8,12,13,24,25], and then for the coupled Schrödinger systems with cubic nonlinearities [1–3,9]. Surprisingly little is known about the system (1.4) with the constraint

$$\int_{\mathbb{R}^N} u^2\, dx = a_1 \quad \text{and} \quad \int_{\mathbb{R}^N} v^2\, dx = a_2 \tag{1.5}$$