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Contractive barycentric maps and L^1 ergodic theorems on the cone of positive definite matrices

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A B S T R A C T

We are concerned with contractive (with respect to the Wasserstein metric) barycenters of probability measures with bounded support on the convex cone of positive definite matrices equipped with the Thompson metric. Based on the important construction schemes of multivariate matrix means, namely the proximal average, and the Cartan mean (the least squares average) for the Cartan–Hadamard metric, we construct a one parameter family of contractive barycentric maps interpolating continuously and monotonically the harmonic, arithmetic and Cartan barycenters. We show that each contractive barycentric map is monotonic for the stochastic order induced by the cone and establish stochastic approximations and L^1 ergodic theorems for the parameterized contractive barycenters.

1. Introduction

A complete metric space (M, d) with a *contractive barycentric map* $\beta : \mathcal{P}^p(M) \to M$ for the Wasserstein distance (alternatively Kantorovich–Rubinstein distance) on its probability measure space of finite pth moments has attracted increasing attention in various fields of pure and applied areas, particularly in geometric analysis, probability measure theory and optimal transport theory [7,8,36,37]. A metric space equipped with a contractive barycentric map is called a *barycentric metric space* [29,31] and plays a fundamental role in the theory of integrations (random variables, expectations and variances), law of large numbers, Birkhoff ergodic theorem, Jensen's inequality [6,10,34,22,30], stochastic generalization of Lipschitz retractions and extension problems of Lipschitz and Hölder maps [29,23,31] and optimal transport theory on Riemannian manifolds [32,33], to cite only a few. In [34], K.-T. Sturm developed a theory of barycenters of probability measures for metric spaces of nonpositive curvature, particularly that class of metric spaces known as Hadamard spaces (alternatively CAT(0) or NPC spaces).

This paper is concerned with construction schemes of contractive barycentric maps of probability measures with bounded support on the Banach Finsler manifold $\mathbb{P}$ of positive definite matrices of fixed

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size, the Finsler structure being derived from the operator norm. The Finsler distance is given by $d(A,B) = ||\log A^{-1/2}BA^{-1/2}||$, where $||\cdot||$ is the operator norm, and coincides with the Thompson metric, a common metric for open normal cones of real Banach spaces. It is invariant under congruence transformations and matrix inversion.

The first successful construction originates from the resolvent average. Bauschke, Moffat and Wang [4] proposed a very interesting family of multivariate matrix means, called the parameterized resolvent mean. The origin of the resolvent mean comes from a convex analysis and optimization context, the proximal average of proper convex lower semicontinuous functions, which is a wonderful and fascinating notion of average of convex functions in the context of convex analysis and optimization (see [2,3]). We show that the parameterized resolvent average can be extended contractively and monotonically to the space of L^∞-probability measures.

The convex cone $\mathbb{P}$ also carries the structure of a Riemannian manifold with distance given by the trace metric $\delta(A,B) = ||\log A^{-1/2}BA^{-1/2}||_2$. More precisely, it is a Cartan–Hadamard Riemannian manifold, a simply connected complete Riemannian manifold with nonpositive sectional curvature (the canonical 2-tensor is non-negative). By Sturm's result in [34], there exists an important contractive barycenter arising as the solution of a least squares problem; for $\mu \in \mathcal{P}^1(\mathbb{P})$ the space of integrable Borel probability measures on $\mathbb{P}$, the Cartan barycenter $\Lambda(\mu)$ of μ is defined as the unique minimizer $\Lambda(\mu) = \underset{Z\in\mathbb{P}}{\arg\min} \int_{\mathbb{P}}[\delta^2(Z,A) - \delta^2(X,A)]d\mu(A)$. (This is independent of X.) Its fundamental contraction property shown by Sturm is $\delta(\Lambda(\mu),\Lambda(\nu)) \le \delta^W(\mu,\nu)$ for the Wasserstein distance δ^W on $\mathcal{P}^1(\mathbb{P})$. For the particular case that $\mu = \sum_{j=1}^n (1/n)\delta_{A_j}$ is a finitely supported uniform probability measure, the Cartan mean $\Lambda(\mu)$ has been studied extensively the past several years by many authors as a multivariable extension of the two variable matrix geometric mean, the δ-geodesic midpoint between two points (see [5,19]). Despite big difference between the δ and d-Wasserstein distances on $\mathcal{P}^1(\mathbb{P})$, the Cartan barycentric map on $\mathcal{P}^1(\mathbb{P})$ is also contractive for the d-Wasserstein distance [22]. We develop a general contraction theorem for barycenters based on monotonic and homogeneous matrix means (Corollary 3.8). Several examples of contractive barycentric maps including the Cartan and resolvent barycentric maps are presented in section 4.

The main result of present paper is a construction of d-contractive barycentric maps $\Lambda_t, t \in [-\infty,\infty]$, on the probability measure space $\mathcal{P}^\infty(\mathbb{P})$ with bounded support on $\mathbb{P}$. This includes the Cartan barycenter at $t = 0$ and also the harmonic and arithmetic barycenters at $t = \pm\infty$, respectively. Indeed, the contractive barycenters Λ_t interpolate *continuously and monotonically* the harmonic, arithmetic and Cartan barycenters over $t \in [-\infty,\infty]$. The main idea is the resolvent average of the Cartan mean for finitely and uniformly supported probability measures

$$\Lambda_t(A_1,\dots,A_n) := \Lambda(A_1 + tI,\dots,A_n + tI) - tI, \quad (t \ge 0)$$

and $\Lambda_t(A_1,\dots,A_n) := \Lambda_{-t}(A_1^{-1},\dots,A_n^{-1})^{-1}$ for $t < 0$, and its *monotonicity for parameters* $\Lambda_t \le \Lambda_s$ for $t \le s$. Its nonexpansive property for the Thompson metric depends on monotonicity for both parameters and variables, and also on homogeneity, which is closely connected to the Thompson metric (but not the Riemannian trace metric δ). We establish that each Λ_t can be monotonically and contractively extended to the Wasserstein space $\mathcal{P}^\infty(\mathbb{P})$: $\Lambda_t(\mu) \le \Lambda_s(\mu)$ for $-\infty \le t \le s \le \infty$ and the limiting property with the arithmetic and harmonic barycentric maps: $\mathcal{H}(\mu) = \lim_{t\to-\infty}\Lambda_t(\mu) \le \Lambda(\mu) \le \lim_{t\to\infty}\Lambda_t(\mu) = \mathcal{A}(\mu)$.

In section 5, we provide detailed proofs with general multivariate matrix means satisfying homogeneity, joint concavity and monotonicity. This in particular provides plenty of contractive barycentric maps of probability measures on the Thompson metric space, in contrast to the Riemannian trace manifold where the Cartan barycenter appeared as the classical example. In section 7 we develop expectations of random variables from the parameterized contractive maps Λ_t and provide a stochastic approximation of $\Lambda_t(\mu)$, evidence there may be a strong law of large numbers based on the derived expectations. In section 8, we establish an L^1 ergodic theorem for Λ_t with $t \in \mathbb{R}$. Our results obtained in this paper are apparently new

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