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Scaling of spectra of self-similar measures with consecutive digits ☆

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A R T I C L E I N F O

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A B S T R A C T

We consider the equally-weighted Cantor measures $\mu_{p,q}$ generated by the iterated function system (IFS) $\{f_i(x) = \frac{x}{p} + \frac{i}{q}\}_{i=0}^{q-1}$, where $2 \leq q \in \mathbb{Z}$ and $q < p \in \mathbb{R}$. It is known that if q divides p, then $\mu_{p,q}$ is a spectral measure with a spectrum

$$\Lambda_{p,q} = \{0, 1, \ldots, q-1\} + p\{0, 1, \ldots, q-1\} + p^2\{0, 1, \ldots, q-1\} + \cdots \text{ (finite sum)}$$

(Dai, He and Lai (2013) [6]). In this paper the authors study some positive integers b such that $b\Lambda_{p,q}$ is also a spectrum of $\mu_{p,q}$.

1. Introduction

A probability measure μ on $\mathbb{R}$ with compact support is said to be a *spectral measure* if there exists a countable set Λ of real numbers, called a spectrum of μ, such that $E(\Lambda) = \{e^{-2\pi i\lambda x} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. A well-known classic example is the Lebesgue measure on $[0, 1]$ for which the set $\mathbb{Z}$ is the only spectrum containing 0. The existence of a spectrum for a probability measure is one of fundamental problems in applied harmonic analysis builded on a measure, and it was initiated by Fuglede in his seminal paper [16]. In 1998, Jorgensen and Pedersen [22] discovered the first families of non-atomic singular spectral measures. They showed that the Bernoulli convolutions are spectral measures if the contraction ratios are the reciprocal of an even integer. Recently, Dai [4] showed that the only spectral Bernoulli convolutions are of the above cases. The details on the background of Bernoulli convolutions and recent topics are given in [4,15,27,29,28] and the references therein. Actually, the spectral measure problem attracts more attention due to Jorgensen and Pedersen's examples. Following this discovery, various singular

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spectral measure on self-similar/self-affine/moran fractal sets have been constructed (see [22,4,6,7,24,26,11, 1,2,17] and references therein). Usually, the following two types of questions have been considered:

(Q1) When is a Borel probability measure μ spectral?

Until now, there are only a few classes of singular spectral measures that are known. It is still a basic problem to find more spectral measures.

(Q2) For a given spectral measure μ, can we find all the spectra of μ?

It is quite challenging to characterize all the spectra of a given singular spectral measure μ (no example is known with this property). The first attempt of the classification of spectra was studied by Dutkay, Han and Sun [8]. They gave a complete characterization of the maximal orthogonal sets of the one-fourth standard Cantor measure (denoted by μ_4) by introducing a labeling tool on the infinite binary tree. They gave some sufficient conditions for a maximal orthogonal set to be a spectrum. Later, Dai, He and Lai [6] gave some sufficient conditions and necessary conditions for a maximal orthogonal set of μ_4 to be a spectrum. Generally speaking, for a given singular spectral measure μ, there are two basic problems (call them spectral eigenvalue problems) as follows:

Case I. Let Λ be a spectrum of μ. Find all real numbers b such that $b\Lambda$ is also a spectrum of μ.

Case II. Find all real numbers b for which there exists a set Λ such that both Λ and $b\Lambda$ are spectra of μ.

Let an iterated function system (IFS) on $\mathbb{R}$ of the form $f_i(x) = \frac{x}{p} + \frac{i}{q}, i = 0, 1, \ldots, q-1$ where $2 \le q \in \mathbb{Z}$ and $p \in \mathbb{R}$. By Hutchinson's theorem [3,13,18], there exists a unique Borel probability measure $\mu_{p,q}$ with compact support $T_{p,q}$ satisfying that

$$\mu_{p,q}(E) = \sum_{i=0}^{q-1} \frac{1}{q} \mu_{p,q}(f_i^{-1}(E)) \tag{1.1}$$

for any Borel set $E \subseteq \mathbb{R}$ and $T_{p,q}$ is the unique compact set satisfying that

$$T_{p,q} = \bigcup_{i=0}^{q-1} f_i(T_{p,q}).$$

When $q = 2$, $\mu_{p,q}$ becomes the standard Cantor measure of contraction ratio $\frac{1}{p}$ denoted by μ_p, i.e., the Bernoulli convolutions. For the $\mu_{p,q}$, there are some known results focused on the above two questions:

- $\mu_{p,q}$ is a spectral measure if and only if $\frac{p}{q} \in \mathbb{Z}$ [7]. If $p = q$, the measure $\mu_{p,q}$ is the Lebesgue measure restricted to the interval $[0, 1]$.
- Let b be a real number. If $p > q$ with $\frac{p}{q} \in \mathbb{Z}$, then

$$b \in \{r \in \mathbb{R} : \text{there exists a set } \Lambda \text{ such that both } \Lambda \text{ and } r\Lambda \text{ are spectra of } \mu_{p,q}\}$$

  if and only if $b = \frac{b_1}{b_2}$, where $\gcd(b_1, b_2) = 1$ and b_1, b_2 are coprime with q respectively [14].

The remaining relatively tractable situation is **Case II** of the above (Q2) for the measure $\mu_{p,q}$. The special cases are the Bernoulli convolution μ_{2k} with $k \in \mathbb{Z}^+$, i.e., the set of all positive integers. It is known that the simplest spectrum for μ_{2k} [22] is

$$\Lambda_{2k} = \left\{ \sum_{j=0}^{n} a_j (2k)^j : \ a_j \in \{0, 1\}, n \in \mathbb{N} := \{0, 1, 2, \ldots\} \right\}.$$

Later, Laba, Wang, Jorgensen, Dutkay and Li et al. investigated for what $b \in \mathbb{N}$, the scaling set $b\Lambda_4$ or $b\Lambda_{2k}$ is also a spectrum of μ_4, or μ_{2k} respectively [23,10,12,20,25]. In this paper, we will investigate those problems for the general case $\mu_{p,q}$.