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Polynomial partition asymptotics

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ABSTRACT

Let $f \in \mathbb{Z}[y]$ be a polynomial such that $f(\mathbb{N}) \subseteq \mathbb{N}$, and let $p_{\mathcal{A}_f}(n)$ denote number of partitions of n whose parts lie in the set $\mathcal{A}_f := \{f(n) : n \in \mathbb{N}\}$. Under hypotheses on the roots of $f - f(0)$, we use the Hardy–Littlewood circle method, a polylogarithm identity, and the Matsumoto–Weng zeta function to derive asymptotic formulae for $p_{\mathcal{A}_f}(n)$ as n tends to infinity. This generalises asymptotic formulae for the number of partitions into perfect dth powers, established by Vaughan for $d = 2$, and Gafni for the case $d \geq 2$, in 2015 and 2016 respectively.

1. Introduction and preliminaries

A partition of a positive integer n is a non-decreasing sequence of positive integers whose sum is n. Let $\mathcal{A} \subseteq \mathbb{N}$ and $p_{\mathcal{A}}(n)$ denote the number of partitions of n such that each part of the partition is restricted to be an element of $\mathcal{A}$. When $\mathcal{A} := \mathbb{N}$, we obtain the well studied unrestricted partition function, usually denoted by $p(n)$. Let $f \in \mathbb{Z}[y]$ be a polynomial such that $f(\mathbb{N}) \subseteq \mathbb{N}$. Then we define $p_{\mathcal{A}_f}(n)$ to be the number of partitions of n whose parts lie in the set $\mathcal{A}_f := \{f(n) : n \in \mathbb{N}\}$. Under mild hypotheses on f, we derive an asymptotic formula for $p_{\mathcal{A}_f}(n)$ using the Hardy–Littlewood circle method and a fine analysis of the Matsumoto–Weng zeta function [5].

In 1918, Hardy and Ramanujan initiated the analytic study of $p(n)$ with the use of the celebrated Hardy–Littlewood circle method [4]. They proved

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}} \quad \text{as} \quad n \to \infty.$$

For fixed $k \geq 2$ they also conjectured an asymptotic formula for the restricted partition function $p_{\mathcal{A}_k}(n)$, where $\mathcal{A}_k$ denotes the set of perfect kth powers. Later in 1934, Wright [15] provided proof for Hardy and

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Ramanujan's conjectured formula concerning $p_{\mathcal{A}_k}(n)$. However, Wright's proof relied heavily on a transformation for the generating function for the sequence $\{p_{\mathcal{A}_k}(n)\}$ that involved generalised Bessel functions.

Vaughan has recently established a simplified asymptotic formula for $p_{\mathcal{A}_k}(n)$ in the case $k = 2$ [14]. This was subsequently generalised for all $k \geq 2$ by Gafni [3]. Using the ideas from [14] and [3], Berndt, Malik, and Zaharescu in [2] have derived an asymptotic formula for restricted partitions in which each part is a kth power in an arithmetic progression. More precisely, for fixed $a_0, b_0, k \in \mathbb{N}$ with $(a_0, b_0) = 1$, they give an asymptotic for $p_{\mathcal{A}_k(a_0,b_0)}(n)$, as n tends to infinity, where $\mathcal{A}_k(a_0, b_0) := \{m^k : m \equiv a_0 \bmod b_0\}$. It is at the end of Berndt, Malik, and Zaharescu's paper [2] that they pose the question of establishing an asymptotic formula for $p_{\mathcal{A}_f}(n)$. To this end, we will follow the implementation of the circle method presented in [2,3,14], with some key innovations. The first is a careful analysis of the Matsumoto–Weng zeta function and the application of a polylogarithm identity to extract the main terms of the asymptotic occurring in Theorem 1.1. For this see Lemma 2.2. The second key innovation is a generalisation of the classical major arc estimate for Waring's problem, see Lemma 2.3.

For other types of formulae for restricted partitions, we refer the reader to [8] and [9]. Interestingly, Vaughan has obtained an asymptotic formula for the number of partitions into primes [13].

We now introduce some notation and preliminaries that will allow us to state Theorem 1.1. Let $d \geq 2$ and suppose

$$f(y) := \sum_{j=0}^{d} a_j y^j \in \mathbb{Z}[y]$$

is fixed such that $(a_0, \ldots, a_d) = 1$ and

$$f(y) - a_0 = a_d y \prod_{j=1}^{d-1} (y + \alpha_j)$$

is such that $\alpha_j \in \mathbb{C} \setminus \mathbb{R}_{\leq -1}$. By convention, for $z \in \mathbb{C}$ we let $-\pi < \arg(z) \leq \pi$ and $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_{d-1}, 0)$. Note that the greatest common divisor condition imposed above is important because it ensures there are no congruence obstructions to representing an integer n with a partition whose parts are values of f.

Evaluations of the Matsumoto–Weng zeta function at integers and residues of its poles naturally appear in our asymptotic formulae for $p_{\mathcal{A}_f}(n)$. We will provide some brief background on this function. Matsumoto and Weng [5] introduced the following r-tuple zeta function

$$\zeta_r\big((s_1, \ldots, s_r); (\beta_1, \ldots, \beta_r)\big) := \sum_{n=1}^{\infty} \frac{1}{(n + \beta_1)^{s_1} \cdots (n + \beta_r)^{s_r}} \tag{1.1}$$

where the $s_j \in \mathbb{C}$ are complex variables and $\beta_j \in \mathbb{C} \setminus \mathbb{R}_{\leq -1}$ for all $1 \leq j \leq r$. Here

$$(n + \beta_j)^{s_j} = \exp(-s_j \log(n + \beta_j))$$

where the branch of the logarithm is fixed as $-\pi < \arg(n + \beta_j) \leq \pi$. This series is clearly well defined and absolutely convergent in the region,

$$\mathrm{Re}(s_1 + \cdots + s_r) > 1.$$

By means of the classical Mellin–Barnes integral formula [5, Eqn. 4], $\zeta_r(\cdot, \boldsymbol{\beta})$ has meromorphic continuation to $\mathbb{C}^r$ with respect to the variables $s_1, \ldots, s_r$ when $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{r-1}, 0)$. One can see [5, Prop. 1] for more details. We will use the one-variable specialisation $s := s_1 = \cdots = s_r$ of (1.1), and its corresponding