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A generalized modified Bessel function and a higher level analogue of the theta transformation formula ☆

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ABSTRACT

A new generalization of the modified Bessel function of the second kind $K_z(x)$ is studied. Elegant series and integral representations, a differential-difference equation and asymptotic expansions are obtained for it thereby anticipating a rich theory that it may possess. The motivation behind introducing this generalization is to have a function which gives a new pair of functions reciprocal in the Koshliakov kernel $\cos(\pi z)\, M_{2z}(4\sqrt{x}) - \sin(\pi z)\, J_{2z}(4\sqrt{x})$ and which subsumes the self-reciprocal pair involving $K_z(x)$. Its application towards finding modular-type transformations of the form $F(z, w, \alpha) = F(z, iw, \beta)$, where $\alpha\beta = 1$, is given. As an example, we obtain a beautiful generalization of a famous formula of Ramanujan and Guinand equivalent to the functional equation of a non-holomorphic Eisenstein series on $SL_2(\mathbb{Z})$. This generalization can be considered as a higher level analogue of the general theta transformation formula. We then use it to evaluate an integral involving the Riemann Ξ-function and consisting of a sum of products of two confluent hypergeometric functions.

1. Introduction

Bessel functions are among the most important special functions of mathematics. Just in mathematics, they encompass differential equations, integral transforms, number theory (especially analytic number theory), Maass forms and mock modular forms, to name a few. The Bessel functions of the first and second kinds of order ν, namely $J_\nu(\lambda)$ and $Y_\nu(\lambda)$, are defined by [34, pp. 40, 64]

$$J_\nu(\lambda) := \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda/2)^{2m+\nu}}{m!\Gamma(m+1+\nu)}, \quad |\lambda| < \infty, \tag{1.1}$$

☆ With an appendix by Nico M. Temme.

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where $\Gamma(s)$ denotes the gamma function, and

$$Y_\nu(\lambda)=\frac{J_\nu(\lambda)\cos(\pi\nu)-J_{-\nu}(\lambda)}{\sin\pi\nu}$$

respectively. The modified Bessel functions of the first and second kinds of order ν are defined by [34, p. 77]

$$I_\nu(\lambda)=\begin{cases}e^{-\frac{1}{2}\pi\nu i}J_\nu(e^{\frac{1}{2}\pi i}\lambda), & \text{if } -\pi<\arg\lambda\le\frac{\pi}{2},\\ e^{\frac{3}{2}\pi\nu i}J_\nu(e^{-\frac{3}{2}\pi i}\lambda), & \text{if } \frac{\pi}{2}<\arg\lambda\le\pi,\end{cases}\tag{1.2}$$

and [34, p. 78]

$$K_\nu(\lambda):=\frac{\pi}{2}\frac{I_{-\nu}(\lambda)-I_\nu(\lambda)}{\sin\nu\pi}.$$

Watson's treatise [34] is a monumental work on Bessel functions and to this day remains a standard reference.

Several generalizations of the Bessel and modified Bessel functions as well have been studied over the years. For these generalizations, we refer the reader to a recent article [24] and the references therein. One of the goals of this paper is to study an interesting new generalization of the modified Bessel function of the second kind. For $z,w\in\mathbb{C}$, $x\in\mathbb{C}\backslash\{x\in\mathbb{R}:x\le 0\}$, and $\mathrm{Re}(s)>\pm\mathrm{Re}(z)$, we define this generalized modified Bessel function by an inverse Mellin transform, namely,

$$K_{z,w}(x):=\frac{1}{2\pi i}\int_{(c)}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right){}_1F_1\left(\frac{s-z}{2};\frac{1}{2};\frac{-w^2}{4}\right){}_1F_1\left(\frac{s+z}{2};\frac{1}{2};\frac{-w^2}{4}\right)2^{s-2}x^{-s}ds,\tag{1.3}$$

where ${}_1F_1(a;c;w)$ is the confluent hypergeometric function defined by [2, p. 188]

$${}_1F_1(a;c;z)=\sum_{n=0}^{\infty}\frac{(a)_nz^n}{(c)_nn!},$$

with $(a)_n$ being the rising factorial $(a)_n:=a(a+1)\cdots(a+n-1)=\Gamma(a+n)/\Gamma(a)$ for $a\in\mathbb{C}$. Here, and throughout the sequel, $\int_{(c)}$ denotes the line integral $\int_{c-i\infty}^{c+i\infty}$.

From (1.3), one property of $K_{z,w}(x)$ follows immediately, namely, that it is an even function in both the variables z and w. When $w=0$, $K_{z,w}(x)$ reduces to the usual modified Bessel function $K_z(x)$ owing to the fact that [26, p. 115, Formula 11.1] for $c=\mathrm{Re}s>\pm\mathrm{Re}\nu$,

$$\frac{1}{2\pi i}\int_{(c)}2^{s-2}\Gamma\left(\frac{s}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{s}{2}+\frac{\nu}{2}\right)x^{-s}\,ds=K_\nu(x).\tag{1.4}$$

The motivation behind the introduction of the generalized modified Bessel function in (1.3) is now explained. The transformation formula for the Jacobian theta function can be put in an equivalent symmetric form [8]

$$\sqrt{\alpha}\left(\frac{1}{2\alpha}-\sum_{n=1}^{\infty}e^{-\pi\alpha^2n^2}\right)=\sqrt{\beta}\left(\frac{1}{2\beta}-\sum_{n=1}^{\infty}e^{-\pi\beta^2n^2}\right),\tag{1.5}$$

where α and β are complex numbers with $\mathrm{Re}(\alpha^2)>0$, $\mathrm{Re}(\beta^2)>0$, and satisfying $\alpha\beta=1$. Either side of this formula is equal to an integral involving the Riemann Ξ-function, that is [8], to

$$\frac{2}{\pi}\int_0^\infty\frac{\Xi(t/2)}{1+t^2}\cos\left(\frac{1}{2}t\log\alpha\right)dt,\tag{1.6}$$