J. Math. Anal. Appl. 459 (2018) 453–463

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Piecewise linear differential system with a center-saddle type singularity

Changwu Zou [a], Jiazhong Yang [b,*]

[a] College of Mathematics and Computer Science, Fuzhou University, Fuzhou, 350116, China
[b] School of Mathematical Sciences, Peking University, Beijing 100871, China

ARTICLE INFO

Article history:
Received 20 September 2016
Available online 27 October 2017
Submitted by Y. Huang

Keywords:
Limit cycle
Piecewise linear differential system
Analytic curve

ABSTRACT

In this paper, we show the existence of piecewise linear differential system with two zones in the plane having the following features: it has a center and a saddle singularity point in each region; the boundary of the zones is an analytic curve; for any given positive integer $n \in \mathbb{N}$, the system has exactly n limit cycles; and all the limit cycles are hyperbolic.

1. Introduction and main results

In recent years, there has been an increasing interest in studying piecewise smooth differential systems in the plane. Quite a number of papers focused on various kinds of systems have been gradually developed. The extensive investigation on this topic can be regarded as an attempt at a general study of discontinuous dynamical systems. The classic monograph [5] tells in details the history of the subject and offers a large amount of related information and broad exploration.

Certainly, it is a very hard nut to crack for a general study of nonlinear systems, let alone the piecewise (nonlinear) ones, since the dynamical properties can be extremely sophisticated. Therefore people somehow have to turn to piecewise linear systems (PWLS). One special class of PWLS catching more attention than others is the so-called piecewise linear systems with two zones (PWLS2). More precisely, people study those systems which have a boundary dividing the whole plane into two regions. One of the convincing reasons why such a study attracts more interest than other systems perhaps lies in the fact that it is logically the simplest case.

When we narrow down our research to PWLS2, one can see that even in this case the phenomena still remains very rich. In fact, when we walk around in such a setting, we are at least facing, among many others,

* Corresponding author.
E-mail addresses: zouchw@pku.edu.cn (C. Zou), jyang@math.pku.edu.cn (J. Yang).

https://doi.org/10.1016/j.jmaa.2017.10.043

the following important factors: (i) The boundary of the separation, and (ii) the type of singularities. More details of these two factors on the dynamics of a given system are discussed below.

As it is well-known that among many dynamical properties, the study of limit cycles, especially the maximal possible number of limit cycles, is always one of the highlights of ordinary differential systems. This is no exception for PWLS2. Now let us see how the above two factors exert influence on the number of limit cycles. A brief survey about this topic is as follows: First of all, for a study of bifurcation of limit cycles of PWLS, we refer the reader to, for example [10,8,6,11,4,15]. As to the number of limit cycles of PWLS, to get more limit cycles, an efficient idea was developed in [12,14], where the authors by dividing the plane into more pieces of regions succeed in obtaining more limit cycles. When restricted to PWLS2, the configuration of limit cycles is as follows. When the separation boundary is a straight line, it is conjectured in [8] that a PWLS2 can have at most two limit cycles. This record was soon broken as Huan and Yang [9], Braga and Mello [1] provided examples with three limit cycles. See also [13] and [7]. When people try to replace the straight line by a curve, the configuration immediately becomes delicate, as it is realized that PWLS2 can generate more than three limit cycles, see for example [2,16,3]. In particular, Braga and Mello [3] constructed a PWLS2 which has n hyperbolic limit cycles for any given integer n. Notice that the separation boundary, whose points are of sewing type except the unique equilibrium point, is a non-smooth polygonal line.

On the other hand, notice that the systems in all these three papers are PWLS2s having two focus singular points. That is, there is a focus type singular point in each region. For convenience, we call such a pair of singular points of the focus–focus type. Indeed, up to now, the focus–focus type singularities are a particular favorite of research. Since the types of the singularities clearly have both global and local influence on the dynamics, therefore to study other kinds of singularities becomes natural and necessary. In particular, to the best of our knowledge, the center-saddle type of systems has caught much less attention.

Concerning the above two aspects, in this paper, we construct a PWLS2 which has exactly n hyperbolic limit cycles for any given integer $n > 0$. The separation boundary of the system constructed is an analytic curve. The system we find is of the center-saddle type. More precisely, we have the following.

Theorem 1. *For any given positive integer $n \in \mathbb{N}$, there exists a piecewise linear system with two zones which has exactly n hyperbolic limit cycles where the separation boundary is an analytic curve.*

To conclude our introduction, we would like to point out that the main theorem relies on the explicit construction of the following system

$$X' = \begin{cases} A^+X - B & y > b\sin x \\ A^-X & y < b\sin x, \end{cases} \tag{1}$$

where $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A^- = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} a \\ 0 \end{pmatrix}$, a and b are real numbers satisfying

$$n\pi < a < (n+1)\pi, \quad 0 < b < \frac{1}{2}(\sqrt{a^2+4} - a). \tag{2}$$

2. Proof of Theorem 1

The proof of the theorem follows from a series of lemmas. First of all, we introduce some notation. Denote by S^0 the curve described by $y = b\sin x$. Then S^0 splits the plane into two open sets

$$U^+ = \{(x,y) \in \mathbb{R}^2 \mid y > b\sin x\} \quad \text{and} \quad U^- = \{(x,y) \in \mathbb{R}^2 \mid y < b\sin x\}. \tag{3}$$