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Multiplicity of large solutions for quasi-monotone pulse-type nonlinearities ☆

Julián López-Gómez [a], Luis Maire [b,*]

[a] Instituto de Matemática Interdisciplinar (IMI), Departamento de Matemática Aplicada, Universidad Complutense de Madrid, 28040, Madrid, Spain
[b] Departamento de Matemática Aplicada, Universidad Complutense de Madrid, Madrid, Spain

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ABSTRACT

This paper discusses the Keller–Osserman condition from a dynamical perspective in order to obtain a rather astonishing multiplicity result of large solutions. It turns out that, for any given increasing positive function $f(u)$ that satisfies the Keller–Osserman condition, destroying the monotonicity of $f(u)$ on a compact set with arbitrarily small measure can originate an arbitrarily large number of explosive solutions. Moreover, some counterexamples to an important result of [6] are given.

1. Introduction

The main goal of this paper is to analyze the existence, uniqueness and multiplicity of positive solutions of the singular boundary value problem

$$\begin{cases} u'' = f(u), & t \in [0,T], \\ u'(0) = 0, \quad u(T) = +\infty, \end{cases} \tag{1.1}$$

where $T \in (0,\infty)$ and $f \in \mathcal{C}^1[0,+\infty)$ satisfies $f(0) = 0$. So, 0 is a constant solution of $u'' = f(u)$. Making a reflection around $t = 0$, these solutions provide us with the positive large solutions of the singular problem

$$\begin{cases} u'' = f(u), & t \in [-T,T], \\ u(-T) = u(T) = +\infty. \end{cases}$$

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* Corresponding author.
E-mail addresses: Lopez_Gomez@mat.ucm.es (J. López-Gómez), luismaire@ucm.com (L. Maire).

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By a positive large solution of (1.1) it is meant any positive solution in $[0,T)$ such that

$$\lim_{t\uparrow T} u(t) = +\infty.$$

Although the singular problem (1.1) has been widely studied in the literature, almost all available results assumed $f \geq 0$ in $[0,\infty)$, or, at least, for sufficiently large u, [3,4]. In this paper we are not imposing any special restriction on the sign of f.

As for arbitrary $f(u)$ the existence and multiplicity of positive solutions of (1.1) might depend on the length of the interval, $T > 0$, it is very natural to analyze the existence of positive explosive solutions of the associated Cauchy problem

$$\begin{cases} u'' = f(u), \\ u(0) = x > 0, \quad u'(0) = 0, \end{cases} \tag{1.2}$$

where $x > 0$ is regarded as parameter.

Even though there is a huge interest in analyzing the existence and the uniqueness of large solutions for wide classes of singular sublinear boundary value problems, because they provide us with the limiting profiles as time grows of the solutions of wide classes of diffusive logistic equations of degenerate type, where the species can growth exponentially in some protection zones of the territory, [1,2,5,8,9,13], and the classical condition of J. B. Keller [10] and R. Osserman [16], as well as some of their variants, as those of [11] and [6], have dominated the scenario of this theory during the last two decades, except in Section 1.1 of [13], no serious effort has been made to realize the true meaning of the several Keller–Osserman conditions involved. Naturally, this pushes experts to back to these conditions from time to time in order to analyze some hidden aspects that were not completely understood before, and [6] is an exceptional example of this.

In most of the literature collected in our bibliography, the Keller–Osserman condition is imposed in order to guarantee the existence of large solutions of some autonomous or non-autonomous problem where the nonlinearity is usually chosen so that the underlying semilinear elliptic equation can exhibit, at most, a unique large solution; the main aim of most of these papers being to show that any large solution must have the same blow-up rate on the edges of the domain to infer from this feature the uniqueness of the large solution by means of a rather standard comparison device. As a consequence of this severe focusing of most of experts's attention, the real meaning of the so called Keller–Osserman condition remains a true enigma!

This prompted us to focus attention in the simplest autonomous one dimensional singular problem (1.1) in order to characterize the values of T for which this singular problem admits a positive solution. Should it be the case, our second aim being either establishing uniqueness, or multiplicity results, keeping in mind, rather crucially, that, in general, (1.1) might admits positive solutions for some range of values of T but not for others. This apparently new methodology, introduced in this paper by the first time, contrasts heavily with most of the available results in the literature, where the Keller–Osserman condition entails the existence of a positive solution of the singular problem (1.1) for every $T > 0$, because the function $f(u)$ is required to satisfy some additional monotonicity property to infer from it the uniqueness of the positive solution of (1.1).

Although there are available some multiplicity results in the context of superlinear indefinite problems, [6,7,12], the most astonishing existing multiplicity results are those of [14] and [15], where it was established that if $a(x)$ changes of sign in the interval $[-T,T]$, then the problem

$$\begin{cases} -u'' = \lambda u - a(x)u^p, \quad \text{in } [-T,T], \\ u(-T) = u(T) = +\infty, \end{cases} \tag{1.3}$$

where $p > 1$, can admit an arbitrarily large number of positive solutions by taking $\lambda < 0$ sufficiently large. In these results the multiplicity is caused by the fact that $a(x)$ changes of sign and $\lambda < 0$ is very large, and