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# On a resolvent estimate for bidomain operators and its applications

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We study bidomain equations that are commonly used as a model to represent the electrophysiological wave propagation in the heart. We prove existence, uniqueness and regularity of a strong solution in $L^p$ spaces. For this purpose we derive an $L^\infty$ resolvent estimate for the bidomain operator by using a contradiction argument based on a blow-up argument. Interpolating with the standard $L^2$-theory, we conclude that bidomain operators generate $C_0$-analytic semigroups in $L^p$ spaces, which leads to construct a strong solution to a bidomain equation in $L^p$ spaces.


## 1. Introduction

The bidomain model is a system related to intra- and extra-cellular electric potentials and some ionic variables. Mathematically, bidomain equations can be written as two partial differential equations coupled with a system of $m$ ordinary differential equations:

$$\partial_t u + f(u,w) - \nabla\cdot(\sigma_i \nabla u_i) = s_i \qquad \text{in } (0,\infty)\times\Omega, \tag{1}$$

$$\partial_t u + f(u,w) + \nabla\cdot(\sigma_e \nabla u_e) = -s_e \qquad \text{in } (0,\infty)\times\Omega, \tag{2}$$

$$\partial_t w + g(u,w) = 0 \qquad \text{in } (0,\infty)\times\Omega, \tag{3}$$

$$u = u_i - u_e \qquad \text{in } (0,\infty)\times\Omega, \tag{4}$$

$$\sigma_i \nabla u_i \cdot n = 0,\ \ \sigma_e \nabla u_e \cdot n = 0 \qquad \text{on } (0,\infty)\times\partial\Omega, \tag{5}$$

$$u(0) = u_0,\ \ w(0) = w_0 \qquad \text{in } \Omega. \tag{6}$$

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Here, functions $u_i$ and $u_e$ are intra- and extra-cellular electric potentials, $u$ is the transmembrane potential (or the action potential) and $w = w(t,x) \in \mathbb{R}^m (m \in \mathbb{N})$ is some ionic variables (current, gating variables, concentrations, etc.). All these functions are unknown. On the other hand, the physical region occupied by the heart $\Omega \subset \mathbb{R}^d$, conductivity matrices $\sigma_{i,e} = \sigma_{i,e}(x)$, external applied current sources $s_{i,e} = s_{i,e}(t,x)$, total transmembrane ionic currents $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ and initial data $u_0$ and $w_0$ are given. The symbol $n$ denotes the unit outward normal vector to $\partial\Omega$. The reader is referred to the books [12] and [21] about mathematical physiology including bidomain models.

There are some literature about well-posedness of bidomain equations. First pioneering work is due to P. Colli-Franzone and G. Savaré [13]. They introduced a variational formulation and derived existence, uniqueness and some regularity results in Hilbert spaces. Here, they assumed nonlinear terms $f, g$ are forms of $f(u,w) = k(u) + \alpha w$, $g(u,w) = -\beta u + \gamma w$ $(\alpha, \beta, \gamma \geq 0)$ with a suitable growth condition on $k$. Examples include cubic-like FitzHugh–Nagumo model, which is the most fundamental electrophysiological model. However, other realistic models cannot be handled by their approach because nonlinear terms are limited. Later M. Veneroni [36] extended to their results by using fixed point argument and established well-posedness of more general and more realistic ionic models. These two papers discussed strong solutions by deriving further regularity of weak solutions. In 2009, Y. Bourgault, Y. Coudiére and C. Pierre [11] showed well-posedness of a strong solution in $L^2$ spaces. They transformed bidomain equations into an abstract evolution equation of the form

$$\begin{cases} \partial_t u + Au + f(u,w) = s, \\ \partial_t w + g(u,w) = 0 \end{cases}$$

by introducing the bidomain operator $A$ in $L^2$ and modified source term $s$. Formally the bidomain operator is the harmonic mean of two elliptic operators, i.e. $(A_i^{-1} + A_e^{-1})^{-1}$ or $A_i(A_i + A_e)^{-1}A_e$, where $A_{i,e}\cdot$ is the elliptic operator $-\nabla \cdot (\sigma_{i,e} \nabla \ \cdot\ )$ with the homogeneous Neumann boundary condition. They proved that the bidomain operator is a non-negative self-adjoint operator by considering corresponding weak formulations. Since their framework is in $L^2$, well-posedness was only proved for $d \leq 3$ in $L^2$ spaces.

The main goal of this paper is to establish $L^p$-theory $(1 < p < \infty)$ and $L^\infty$-theory for the bidomain operator with applications to bidomain equations. More explicitly, we shall prove that the bidomain operator forms an analytic semigroup $e^{-tA}$ both in $L^p$ and $L^\infty$. By this result we are able to construct a strong solution in $L^p$ for any space dimension $d$ (by taking $p$ large if necessary). Our result allows any locally Lipschitz nonlinear terms.

To derive analyticity it is sufficient to derive resolvent estimates. For $L^p$ resolvent estimates a standard way is to use the Agmon's method (e.g. [23], [34]). The main idea of the method is as follows. If we have a $W^{2,p}(\Omega \times \mathbb{R})$ a priori estimate for the operator $A - e^{i\theta}\partial_{tt}$, then $A$ has an $L^p$ resolvent estimate. Unfortunately, it seems difficult to derive such a $W^{2,p}$ a priori estimate because of nonlocal structure of the bidomain operator. Thus we argue in a different way.

We first establish an $L^\infty$ resolvent estimate for the bidomain operator by a contradiction argument including a blow-up argument. We then derive an $L^p$ resolvent estimate for $2 \leq p \leq \infty$ by interpolating $L^2$ and $L^\infty$ results. The $L^p$-theory for $1 < p < 2$ is established by a duality argument. Note that a standard idea to derive an $L^\infty$ resolvent estimate due to Masuda–Stewart (see the third next paragraph) does not apply because their method is based on an $L^p$ resolvent estimate, which we would like to prove.

A blow-up argument was first introduced by E. De Giorgi [14] in order to study regularity of a minimal surface. It is also efficient to derive a priori estimates for solutions of a semilinear elliptic problem [16] and a semilinear parabolic problem [17], [18]. Recently, K. Abe and the first author [1], [2] showed that the Stokes operator is a generator of an analytic semigroup on $C_{0,\sigma}(\Omega)$, the $L^\infty$-closure of $C_{c,\sigma}^\infty(\Omega)$ (the space of smooth solenoidal vector fields with compact support in $\Omega$) for some class of domains $\Omega$ including bounded and exterior domains by using a blow-up argument for a nonstationary problem. For a direct proof extending

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