J. Math. Anal. Appl. 459 (2018) 590–603

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The existence of Fourier basis for some Moran measures ☆

Feng-Li Yin, Meng Zhu *

School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, PR China

ARTICLE INFO

Article history:
Received 13 August 2017
Available online 23 October 2017
Submitted by M. Laczkovich

Keywords:
Spectral measures
Moran iterated function systems
Moran measures
Spectrum

ABSTRACT

Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of integers bigger than 1 and let $\{\mathcal{D}_n\}_{n=1}^{\infty}$ be a sequence of digit sets in $\mathbb{Z}$ where $\mathcal{D}_n = \{0, r_n, 2r_n, \cdots, (q_n-1)r_n\}$. The family of functions $\{f_{n,d}(x) = b_n^{-1}(x+d) : d \in \mathcal{D}_n\}_{n=1}^{\infty}$ is called a *Moran iterated function system (IFS)*. In this paper we prove that the associated Moran measure generated by an infinite convolution of atomic measures with equal distribution

$$\mu_{\{b_n\},\{\mathcal{D}_n\}} = \delta_{b_1^{-1}\mathcal{D}_1} * \delta_{(b_1b_2)^{-1}\mathcal{D}_2} * \cdots * \delta_{(b_1b_2\cdots b_n)^{-1}\mathcal{D}_n} * \cdots$$

is a spectral measure if $r_n q_n | b_n$.

Contents

1. Introduction

As is well known the exponential functions $\{e^{2\pi i<\lambda,x>} : \lambda \in \mathbb{Z}^d\}$ form an Fourier basis for $L^2([0,1]^d)$ and it is now one of the fundamental pillars in modern mathematics. It is natural to ask what other measures have this property, that there is a family of exponential functions which form an orthonormal basis for their L^2-space?

☆ This work was supported by the National Natural Science Foundation of China 11271148. School of Mathematics and Statistics and Hubei Key Laboratory of Mathematical Sciences, CCNU.

* Corresponding author.
E-mail addresses: yinfengli05181@163.com (F.-L. Yin), zhumengsx86@163.com (M. Zhu).

https://doi.org/10.1016/j.jmaa.2017.09.039
0022-247X/

In 1998, Jorgensen and Pedersen [16] made a surprising discovery: they constructed a fractal measure μ on a Cantor set which admits an orthonormal basis for $L^2(\mu)$. This opened up a new field in researching the orthogonal harmonic analysis of fractal measures including self-similar/self-affine measures and general Moran measures. In [24], Strichartz proved the surprising result that the Fourier series for the Jorgensen–Pedersen example have much better convergence properties than their classical counterparts on the unit interval.

Definition 1.1. Let μ be a compactly supported Borel probability measure on $\mathbb{R}$. We say that μ is a *spectral measure* if there exists a countable set Λ of $\mathbb{R}$ such that $E(\Lambda) := \left\{e^{2\pi i\lambda x} : \lambda \in \Lambda\right\}$ forms an orthonormal basis for $L^2(\mu)$. In this case, Λ is called a *spectrum* of μ and (μ, Λ) is called a *spectral pair*.

Later on, more fractal spectral measures were constructed, even in higher dimensions [4–7,9,10,12,13,15, 17–21,23]. However, most of these fractal measures were generated by self-affine *iterated function systems* (IFSs) [11]. In this paper, we will study measures generated by *Moran IFSs* (see Definition 1.2), which are generalizations of self-affine IFSs.

Definition 1.2. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of integers with all $b_n \geqslant 2$ and let $\{\mathcal{D}_n\}_{n=1}^{\infty}$ be a sequence of digit sets with $0 \in \mathcal{D}_n \subset \mathbb{Z}$ for each $n \geqslant 1$. We call the function system $\{f_{n,d}(x) = b_n^{-1}(x + d) : d \in \mathcal{D}_n\}_{n=1}^{\infty}$ a *Moran IFS*.

If $\sup\{x : x \in b_n^{-1}\mathcal{D}_n, n \geqslant 1\} < \infty$, then there exists a Borel probability measure with compact support defined by the convolution

$$\mu_{\{b_n\},\{\mathcal{D}_n\}} = \delta_{b_1^{-1}\mathcal{D}_1} * \delta_{(b_1b_2)^{-1}\mathcal{D}_2} * \cdots ,$$

where $\delta_{rE} = \frac{1}{\#E}\sum_{a\in E}\delta_{ra}$ ($\#E$ is the cardinality of E) and δ_{ra} is the Dirac measure at ra, the sign $*$ means the convolution and the convergence is in weak sense. In this case, $\mu_{\{b_n\},\{\mathcal{D}_n\}}$ is called a *Moran measure*, and its support is the *Moran set*

$$T = \sum_{n=1}^{\infty}(b_1b_2\cdots b_n)^{-1}\mathcal{D}_n = \left\{\sum_{n=1}^{\infty}(b_1b_2\cdots b_n)^{-1}d_n : d_n \in \mathcal{D}_n, n \geq 1\right\}.$$

Moran sets and Moran measures appear frequently in dynamic systems, multifractal analysis and geometry number theory (see [12]), etc. Until now, there are only a few results on the spectrality of Moran measures [1–3,14].

The main question addressed in spectral measure theory is the following:

Question. When are the above Moran measures $\mu_{\{b_n\},\{\mathcal{D}_n\}}$ spectral?

The following is our partial answer to this question.

Theorem 1.3. *Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of integers bigger than 1, and let $\{\mathcal{D}_n\}_{n=1}^{\infty}$ be a sequence of digit sets with $\mathcal{D}_n = \{0, r_n, 2r_n, \cdots, (q_n - 1)r_n\}$ in $\mathbb{Z}$. Then the associated Moran measure*

$$\mu_{\{b_n\},\{\mathcal{D}_n\}} = \delta_{b_1^{-1}\mathcal{D}_1} * \delta_{(b_1b_2)^{-1}\mathcal{D}_2} * \cdots * \delta_{(b_1b_2\cdots b_n)^{-1}\mathcal{D}_n} * \cdots$$

is a spectral measure if $r_nq_n|b_n$.

Corollary 1.4. *Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of integers bigger than 1, and let $\{\mathcal{D}_n\}_{n=1}^{\infty}$ be a sequence of digit sets with $\mathcal{D}_n = r\{0, 1, 2, \cdots, (q_n - 1)\}$ in $\mathbb{Z}$. Then the associated Moran measure*