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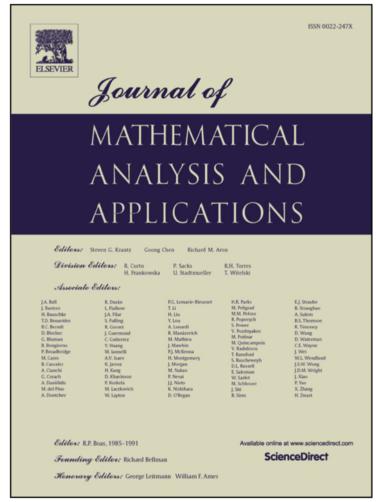

A support theorem for generalized convexity and its applications

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A SUPPORT THEOREM FOR GENERALIZED CONVEXITY AND ITS APPLICATIONS.

ANDRZEJ OLBRYŚ

Abstract. In the present paper we introduce a notion of (ω, t)-convexity as a natural generalization of the notion of usual t-convexity, t-strongly convexity, approximate t-convexity, delta t-convexity and many other. The main result of this paper establishes the necessary and sufficient conditions under which an (ω, t)- convex map can be supported at a given point by an (ω, t)-affine support function. Several applications of this support theorem are presented. For instance, new characterizations of inner product spaces among normed spaces involving the notion of (ω, t)-convexity are given.

1. Introduction and terminology

Let $t \in (0, 1)$ be a fixed number and let $\mathbb{Q}(t)$ be the smallest field containing the singleton $\{t\}$. Throughout the whole paper (unless explicitly stated otherwise) X denotes a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and D stands for a non-empty t-convex set i.e.

$$tD + (1-t)D \subseteq D.$$

Now, for a given function $\omega : D \times D \times [0, 1] \to \mathbb{R}$ we introduce a notion of (ω, t)-*convexity.* A function $f : D \to \mathbb{R}$ is said to be:

(ω, t)-*convex*, if

$$f(tx + (1-t)z) \leq tf(x) + (1-t)f(z) + \omega(x, z, t), \quad x, z \in D,$$

(ω, t)-*concave*, if

$$tf(x) + (1-t)f(z) + \omega(x, z, t) \leq f(tx + (1-t)z), \quad x, z \in D.$$

If f is at the same time (ω, t)-*convex* and (ω, t)-*concave* then we say that it is an (ω, t)-*affine.* In this case f satisfies the following functional equation

$$tf(x) + (1-t)f(z) + \omega(x, z, t) = f(tx + (1-t)z), \quad x, z \in D.$$

If $t = \frac{1}{2}$ then f is said to be ω-*midpoint convex* (ω-*midpoint concave*, ω-*midpoint affine*). If the above inequalities are satisfied for all numbers $t \in [0, 1]$ (where D stands for a convex set) then we say that f is ω-*convex* (ω-*concave*, ω-*affine*, respectively).

The notion of ω-*convexity* is a common generalization of the notion of usual *convexity, strong- convexity, approximate-convexity, delta-convexity* and many other. The term on the left-hand side of the inequality is the same in all definitions while the right-hand side of all inequalities has different form.

Let $(X, \|\cdot\|)$ be a real normed space, D be a convex subset of X and let $c > 0$. A function $f : D \to \mathbb{R}$ is *called strongly t-convex* ($t \in (0, 1)$) with modulus $c > 0$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2,$$

for all $x, y \in D$. If the above inequality is satisfied with $t = \frac{1}{2}$ then f is said to be *strongly midpoint convex* function. If f is t-*strongly convex* function for all $t \in [0, 1]$ then we say that it is *strongly convex.*

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