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Two symmetry constraints for a generalized Dirac integrable hierarchy

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A B S T R A C T

By a two-by-two matrix spectral problem, a generalized Dirac integrable hierarchy is presented. A Hamiltonian structure of the obtained hierarchy is established by trace identity, and its Liouville integrability is proved. Then, through Bargmann symmetry constraint, spatial part of the Lax pairs and adjoint Lax pairs is nonlinearized as a completely integrable finite-dimensional Hamiltonian system. Next, under an implicit symmetry constraint, both spatial part and temporal parts of the Lax pairs and adjoint Lax pairs are all nonlinearized as completely integrable finite-dimensional Hamiltonian systems. Ultimately, the involutive representation of solution of the generalized Dirac integrable hierarchy is given.

1. Introduction

It is well known that searching new integrable infinite-dimensional Hamiltonian systems is an important and interesting task in mathematical physics [4,10,13–17,19]. A hierarchy of evolution equations

$$u_{t_n} = K_n(u), \tag{1}$$

is called to be integrable in Lax sense, if it can be written as a compatibility condition

$$U_t = V_x^{(n)} - UV^{(n)} + V^{(n)}U = V_x^{(n)} - [U, V^{(n)}] \tag{2}$$

of a suitable spatial spectral problem

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$$\varphi_x = U(u,\lambda)\varphi, \tag{3}$$

and the associated temporal spectral problem

$$\varphi_{t_n} = V^{(n)}(u,\lambda)\varphi, \tag{4}$$

where $u = u(x,t)$ is a vector-valued real function, and $x,t \in R$. λ is a spectral parameter, and $\lambda_t = 0$. The Eq. (3) and Eq. (4) are called a Lax pair of Eq. (1).

One of the important problems in the theory of integrable infinite-dimensional Hamiltonian systems is to search for a Hamiltonian operator J and a hierarchy of conserve functionals $\{\tilde{H}_n\}_{n=0}^{\infty}$ so that the Eq. (1) can be rewritten into the Hamiltonian form

$$u_{t_n} = J\frac{\delta \tilde{H}_n}{\delta u}, n \geq 1, \tag{5}$$

where $\tilde{H}_n = \int H_n dx$, $\delta/\delta u$ stands for the variational derivative. If a hierarchy of evolution equations possesses a Hamiltonian structure and infinitely many conserved functionals which are in involution in pairs with respect to a corresponding Poisson bracket, then the hierarchy of evolution equations is called to be Liouville integrable. When U is in sl(2,R), many integrable Hamiltonian systems in Liouville sense have been presented and studied [16,17]. Recently, matrix spectral problems associated with U in so(3,R) also are researched, the corresponding integrable infinite-dimensional Hamiltonian systems are deduced [13,15,19].

In addition, in the soliton theory, an approach called as nonlinearization method of the Lax pairs of soliton equations has been presented and developed [2,3]. Under the constraint between the potentials and the eigenfunctions, the Lax pairs of soliton equations be nonlinearized into the completely integrable finite-dimensional Hamiltonian systems in the Liouville sense. For many soliton equations, the solutions may be derived by solving the obtained completely integrable finite-dimensional Hamiltonian systems [1]. More recently, the nonlinearization method has been generalized to the binary nonlinearization method, which involves both the Lax pairs and the adjoint Lax pairs for soliton equations and thus the contents of research are much richer and more systematic [6–9,11,12]. This provides a powerful tool for decomposing integrable infinite-dimensional Hamiltonian systems by two completely integrable finite-dimensional Hamiltonian systems in Liouville sense.

We know that the Dirac equation

$$\begin{cases} q_t = -\frac{1}{2}r_{xx} + q^2 r + r^3, \\ r_t = \frac{1}{2}q_{xx} - qr^2 - q^3, \end{cases}$$

is an important soliton equation. For the Dirac equation, a generalized integrable hierarchy and an integrable coupling hierarchy as well as the Darboux transformation of the integrable coupling are studied [5,18,20]. In this paper, we are going to study a generalized Dirac equation and it the corresponding integrable hierarchy.

This paper is organized as follows. In Section 2, we introduce the spectral problem

$$\varphi_x = U(u,\lambda)\varphi,\ U(u,\lambda) = \begin{pmatrix} q & \lambda + r + \frac{1}{2}\alpha(q^2+r^2) \\ -\lambda + r - \frac{1}{2}\alpha(q^2+r^2) & -q \end{pmatrix}, \tag{6}$$

where $u = (q,r)^T$ is the potential vector, $\varphi = (\varphi_1,\varphi_2)^T$ is eigenfunction vector, λ is the spectral parameter and $\lambda_t = 0$. α is an arbitrary constant. When $\alpha = 0$, spectral problem (6) becomes the standard Dirac spectral problem. Therefore, the Eq. (6) is a generalized Dirac spectral problem, starting from the Eq. (6) we present a generalized Dirac integrable hierarchy. In Section 3, we establish a Hamiltonian structure of the

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