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Interpolation formulas with derivatives in de Branges spaces II

Felipe Gonçalves [a,*], Friedrich Littmann [b]

[a] *University of Alberta, Mathematical and Statistical Sciences, 632 Central Academic Building, Edmonton, Alberta, T6G 2G1, Canada*
[b] *Department of mathematics, North Dakota State University, Fargo, ND 58105-5075, United States*

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ABSTRACT

We investigate necessary and sufficient conditions under which entire functions in de Branges spaces can be recovered from function values and values of derivatives. Our main focus is on spaces with a structure function whose logarithmic derivative is bounded in the upper half-plane.

1. Introduction

One of the classical problems in complex analysis is to reconstruct an entire function from a countable set of data. This article considers reconstruction of an entire function F from its values and the values of its derivatives up to a specified order at a discrete set of points on the real line. The entire function F will be an element of a reproducing kernel Hilbert space in the sense of de Branges [5] (we review the pertinent facts of de Branges spaces in Section 2.1). We are interested in reconstruction formulas that converge to F in the norm of the space.

To motivate the results, we briefly describe the situation in the classical Paley–Wiener space. Given $\tau > 0$ the Paley–Wiener space $PW(\tau)$ is defined as the space of entire functions of exponential type at most τ such that their restriction to the real axis belongs to $L^2(\mathbb{R})$. These are special spaces with a reproducing kernel structure. The reproducing kernel of $PW(\tau)$ is given by

$$K(w,z) = \frac{\sin \tau(z-\overline{w})}{\pi(z-\overline{w})}$$

* Corresponding author.
E-mail addresses: felipe.goncalves@ualberta.ca (F. Gonçalves), friedrich.littmann@ndsu.edu (F. Littmann).

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and

$$F(w) = \int_{\mathbb{R}} F(x)\overline{K(w,x)}\mathrm{d}x,$$

for every $F \in PW(\tau)$.

A basic result of the theory of Paley–Wiener spaces is that for all $F \in PW(\tau)$ we have

$$F(z) = \frac{\sin(\tau z)}{\pi} \sum_{n\in\mathbb{Z}} \frac{(-1)^n F(\pi n/\tau)}{z - \pi n/\tau}, \tag{1.1}$$

where the sum converges in $L^2(\mathbb{R})$ as well as uniformly on compact sets of $\mathbb{C}$ (this is sometimes called the Shannon–Whittaker interpolation formula). The existence of interpolation formulas using derivatives is also known in Paley–Wiener spaces. In [28, Theorem 9], J. Vaaler proved that

$$F(z) = \left(\frac{\sin \tau x}{\tau}\right)^2 \sum_{n\in\mathbb{Z}} \left(\frac{F(\pi n/\tau)}{(z-\pi n/\tau)^2} + \frac{F'(\pi n/\tau)}{z - \pi n/\tau}\right), \tag{1.2}$$

for every $F \in PW(2\tau)$.

The Paley–Wiener space is a particular case of a wider class of spaces of entire functions called de Branges spaces (see [5, Chapter 2]). A generalization of Vaaler's result in de Branges spaces was obtained by the first named author in [20] which we describe next. For a given Hermite–Biehler function E, a de Branges space $\mathcal{H}(E^2)$ can be constructed where the intended interpolation formulas will take place. With $E^*(z) = \overline{E(\bar{z})}$ we define $A(z) = 2^{-1}(E(z) + E^*(z))$ and $B(z) = (i/2)(E(z) - E^*(z))$. Furthermore, φ denotes the phase of E, that is, $E(x)e^{i\varphi(x)} \in \mathbb{R}$ for all real x (it can be shown that φ is analytic in a neighborhood of $\mathbb{R}$ and $\varphi'(x) > 0$ for all real x). We denote by $\mathcal{T}_B$ the set of (real) zeros of B.

Throughout this paper we use for f, g in a set $\mathcal{F}$ the notation $f \simeq g$ to mean that there exist positive constants C, D with $Cf \le g \le Df$, and the constants are uniform in $\mathcal{F}$. Similarly, $f \lesssim g$ stands for the statement $f \le Cg$ with uniform $C > 0$.

Theorem A. ([20, Theorem 1]) *Let E be a Hermite–Biehler function such that $\mathcal{H}(E^2)$ is a de Branges space closed under differentiation, $AB \notin \mathcal{H}(E^2)$ and $\varphi'(t) \ge \delta > 0$ for some $\delta > 0$ and all $t \in \mathcal{T}_B$. Then*

$$F(z) = B(z)^2 \sum_{t\in\mathcal{T}_B} \left(\frac{F(t)}{B'(t)^2(z-t)^2} + \frac{F'(t)B'(t) - F(t)B''(t)}{B'(t)^3(z-t)}\right),$$

and the series converges in $\mathcal{H}(E^2)$ as well as uniformly on compact subsets of $\mathbb{C}$.

This result immediately leads to questions regarding necessity and sufficiency of the assumptions under which the conclusions of the theorem hold. Our main results provide answers to most of these questions.

(1) The requirement that $\mathcal{H}(E^2)$ is closed under the differentiation operator is a natural condition, since we use derivatives for reconstruction, but it is usually hard to check directly. A sufficient criterion was given by A. Baranov [1, Theorem 3.2] who showed that if $E'/E \in H^\infty(\mathbb{C}^+)$ (see Section 6), then differentiation defines a bounded operator on $\mathcal{H}(E)$. He also showed that the reverse implication is not true (see [1, Section 4.2]). Our first result connects here, we prove in Theorem 1 that $E'/E \in H^\infty(\mathbb{C}^+)$ is actually equivalent to $\mathcal{H}(E^2)$ being closed under differentiation.