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Firey–Shephard problems for homogeneous measures

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ABSTRACT

We introduce a concept of Firey-projection function for μ-measure, which is an extension of L_p-projection body. We consider the Firey–Shephard problem for measures, and connect it with L_p-projection body. Using this connection, we give a negative answer to the Firey–Shephard problem for homogeneous measures for all $p > 1$, $p \neq n$ and $n \geq 2$. Moreover, we also prove a measure comparison theorem.

1. Introduction

Let $\mathcal{K}^n$ denote the class of convex bodies (compact convex subsets with nonempty interior) in n-dimensional Euclidean space $\mathbb{R}^n$. $\mathcal{K}_o^n$ denotes the subset of convex bodies containing the origin in their interior in $\mathcal{K}^n$, and $\mathcal{K}_e^n$ denotes the subset of origin-symmetric convex bodies in $\mathcal{K}_o^n$.

For a convex body K in $\mathbb{R}^n$, its support function $h_K = h(K, \cdot) : \mathbb{R}^n \to [0, +\infty)$, is defined by $h(K, x) = \max\{x \cdot y : y \in K\}$. For $K, L \in \mathcal{K}^n$ and $\alpha, \beta \geq 0$ (not both zero), we define the Minkowski linear combination $\alpha K + \beta L$ by

$$h(\alpha K + \beta L, u) = \alpha h(K, u) + \beta h(L, u),$$

for $u \in S^{n-1}$. The set sum "+" is called by Minkowski sum.

The Gauss map of K, $\mathcal{V}_K : \partial K \to S^{n-1}$, maps every $x \in \partial K$ to the set of all unit normal vectors of x with respect to K. The surface area measure $S(K, \cdot)$ is the measure on the unit sphere S^{n-1} defined as the push forward to S^{n-1} of the $(n-1)$-dimensional Hausdorff measure on ∂K via the Gauss map $\mathcal{V}_K$. The *projection body of* K, ΠK, is defined by

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$$h_{\Pi K}(\theta) = \mathrm{Vol}_{n-1}(K|\theta^\perp) = \frac{1}{2}\int_{S^{n-1}} |\theta\cdot u| dS(K,u), \tag{1.1}$$

for every $\theta \in S^{n-1}$.

In [18], Shephard asks the following problem. Let K, L be origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$\mathrm{Vol}_{n-1}(K|\theta^\perp) \le \mathrm{Vol}_{n-1}(L|\theta^\perp),$$

for every $\theta \in S^{n-1}$. Does it follow that

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)?$$

It was shown independently by Petty [13] and Schneider [15] that the answer is affirmative if $n \le 2$ and negative if $n \ge 3$. In [8], Koldobsky, Ryabogin and Zvavitch give a Fourier analytic solution to Shephard's problem. See [2,6] for its history and applications.

Considering (1.1), we note that the Shephard problem asks whether

$$\Pi K \subseteq \Pi L$$

implies

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L).$$

In 1962, Firey [1] extended the notion of the Minkowski sum to L_p-sums of convex bodies for every real p, also called by Firey sum. Let $p \ge 1$ and $\alpha, \beta \ge 0$ (not both zero). For $K, L \in \mathcal{K}_o^n$, L_p-combination $\alpha\cdot K +_p \beta\cdot L$ is introduced in [1] by

$$h(\alpha\cdot K +_p \beta\cdot L, u)^p = \alpha h(K,u)^p + \beta h(L,u)^p, \tag{1.2}$$

for $u \in S^{n-1}$. Note that we use "$\cdot$" rather than "$\cdot_p$" to denote Firey scalar multiplication.

After that, Lutwak [10,11] showed that these Firey-sums lead to L_p-Brunn–Minkowski theory, for each $p \ge 1$, which is a great development of the classical Brunn–Minkowski theory. In [10] Lutwak introduced the notion of L_p-surface area measure (see Definition 1.1). The *p-projection body* is given in [11,12] by

$$h(\Pi_p K, \theta)^p = \frac{1}{2n}\int_{S^{n-1}} |\theta\cdot u|^p dS_p(K,u), \qquad \forall\theta \in S^{n-1}. \tag{1.3}$$

As a generalization of the Shephard problem, Ryabogin and Zvavitch [14] considered the following Firey–Shephard problem. Let $p \ge 1$ and let $K, L \in \mathcal{K}^n$ such that

$$\Pi_p K \subseteq \Pi_p L.$$

Does it follow that

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L), \qquad \text{for } 1 \le p < n,$$

and

$$\mathrm{Vol}_n(K) \ge \mathrm{Vol}_n(L), \qquad \text{for } n < p?$$