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Local approximation of non-holomorphic discs in almost complex manifolds

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ABSTRACT

We provide a local approximation result of non-holomorphic discs with small $\bar\partial$ by pseudoholomorphic ones. As an application, we provide a certain gluing construction.

0. Introduction

In [11], J.-P. Rosay stated the following problem for complex manifolds: can a smooth non-holomorphic disc φ with a small $\overline{\partial}\varphi$ always be approximated by a holomorphic one? The question is very general and, in fact, his paper itself contains a counterexample in a compact Riemann surface of genus ≥ 2 (due to L. Lempert). However, under certain restrictions on the initial disc φ the answer turned out to be positive.

In this paper we address the same question but for the case of non-integrable structures. In particular, we give sufficient conditions for such an approximation result to be valid locally in $(\mathbb{R}^{2n}, J)$ (Theorem 5). We stress that, in contrast with the integrable case, a certain uniform bound is imposed on the L^p-norm of the differential $d\varphi$. The proof is based on the implicit function theorem for the linearization of the $\overline{\partial}_J$ operator and a careful study of the existence of a bounded right inverse (Theorem 2).

Finally, motivated by [7,8] we present in the last section an application of the above result. More precisely, we glue together two J-holomorphic halves of an unit disc in order to obtain one holomorphic object.

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1. Preliminaries

Throughout the paper we denote by Δ the open unit disc in $\mathbb{C}$.

1.1. *Almost complex manifolds and pseudoholomorphic discs*

Let M be a real smooth manifold M. An *almost complex structure* J on M is a $(1,1)$ tensor field satisfying $J^2 = -Id$. The pair (M, J) is called an *almost complex manifold*. Let J_{st} be the standard structure on $\mathbb{R}^{2n}$, that is, $(\mathbb{R}^{2n}, J_{st}) \cong \mathbb{C}^n$. A differentiable map $u\colon (M', J') \longrightarrow (M, J)$ between two almost complex manifolds is (J', J)*-holomorphic* if it satisfies

$$J(u(q)) \circ d_q u = d_q u \circ J'(q),$$

for every $q \in M'$, where $d_q u$ denotes the differential map of u at q. When M' is the unit disc Δ, then such a map $u : \Delta \to (M, J)$ is called a J*-holomorphic disc*. Equivalently, u is a J-holomorphic disc whenever the following non-linear operator vanishes

$$\overline{\partial}_J u(v) = \frac{1}{2}\left(du(v) + J(u)du(J_{st}v)\right) = 0.$$

1.2. *The local equation*

Suppose that J is a smooth almost complex structure defined in an open set $U \subseteq \mathbb{R}^{2n}$. Then it may be represented by a $\mathbb{R}$-linear operator $J(z)\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfying $J(z)^2 = -Id$. Further, the J-holomorphy equation for a J-holomorphic disc $u\colon \Delta \to U \subseteq \mathbb{R}^{2n}$ can be written as

$$\frac{\partial u}{\partial y} - J(u)\frac{\partial u}{\partial x} = 0.$$

Moreover, we can rewrite it in its complex form

$$u_{\bar\zeta} + A(u)\overline{u_\zeta} = 0, \tag{1}$$

where $\zeta = x + iy \in \mathbb{C}$ and

$$A(z)(v) = (J_{st} + J(z))^{-1}(J(z) - J_{st})(\bar v)$$

is a complex linear endomorphism for every $z \in U$ and $v \in \mathbb{C}^n$. Hence A can be considered as a $n \times n$ complex matrix of the same regularity as $J(z)$ acting on $v \in \mathbb{C}^n$. We call A the *complex matrix of* J.

Note that the above complex form (1) is valid only when $J(z) + J_{st}$ is invertible. This, in particular, can be achieved locally by a change of coordinates in a neighborhood of any given point [4, Lemma 1] or in a neighborhood of $u(\overline{\Delta})$ where $u\colon \overline{\Delta} \to \mathbb{R}^{2n}$ is an embedded J-holomorphic disc (see the Appendix in [6]); or globally, when J is tamed by the standard symplectic form ω_{st} [1] (see also [13, Proposition 2.8]). We denote by $\mathcal{J}$ the set of all smooth structures on $\mathbb{R}^{2n}$ satisfying such a condition and remark that it is in a one-to-one correspondence with the set of complex matrices A satisfying the condition $\det(I - A\bar{A}) \neq 0$ (see [12]).

1.3. *Sobolev spaces and the Cauchy–Green operator*

Let $p > 2$ and $k \in \mathbb{N}$. Let $\Omega \subset \mathbb{C}$ be bounded. We denote by $L^p(\Omega)$ the classical Lebesgue space and by $W^{k,p}(\Omega)$ the Sobolev space of maps $u\colon \Omega \to \mathbb{C}^n$ whose derivatives up to order k are in $L^p(\Omega)$. We sometimes