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Density solutions to a class of integro-differential equations

Wissem Jedidi [a], Thomas Simon [b,*], Min Wang [b]

[a] Department of Statistics and Operation Research, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
[b] Laboratoire Paul Painlevé, Université Lille 1, Cité Scientifique, F-59655 Villeneuve d'Ascq Cedex, France

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A B S T R A C T

We consider the integro-differential equation $\mathrm{I}^\alpha_{0+} f = x^m f$ on the half-line. We show that there exists a density solution, which is then unique and can be expressed in terms of the Beta distribution, if and only if $m > \alpha$. These density solutions extend the class of generalized one-sided stable distributions introduced in [29] and more recently investigated in [27]. We study various analytical aspects of these densities, and we solve the open problems about infinite divisibility formulated in [27].

1. Introduction and statement of the results

In this paper, we are concerned with the following integro-differential equation

$$x^m f(x) \;=\; \frac{1}{\Gamma(\alpha)} \int_0^x (x-v)^{\alpha-1} f(v)\, dv \tag{1}$$

on $(0, \infty)$, with $\alpha > 0$ and $m \in \mathbb{R}$. This equation can be written in a more compact way as

$$\mathrm{I}^\alpha_{0+} f \;=\; x^m f,$$

where I^α_{0+} is the left-sided Riemann–Liouville fractional integral on the half-axis. We refer to the comprehensive monograph [22] for more details on fractional operators and the corresponding differential equations.

* Corresponding author.
E-mail addresses: wissem_jedidi@yahoo.fr (W. Jedidi), simon@math.univ-lille1.fr (T. Simon), min.wang@math.univ-lille1.fr (M. Wang).

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We are interested in density solutions to (1), that is we are searching for such f satisfying (1) which are also probability densities on $(0,\infty)$. In this framework, the identities (2.1.31) and (2.1.38) in [22] imply that the auxiliary function $h = \mathrm{I}^{\alpha}_{0+}f$ is a solution to the fractional differential equation

$$\mathrm{D}^{\alpha}_{0+}h \;=\; x^{-m}h, \tag{2}$$

where D^{α}_{0+} is the left-sided Riemann–Liouville fractional derivative. This latter equation can be solved in the case $m = 1$ in terms of the classical Wright function – see Theorem 5.10 in [22], and we will briefly come back to this example in Section 3.

Observe that density solutions to (1) may not exist. If $\alpha = m$ for example, then (1) becomes

$$f(x) \;=\; \frac{1}{\Gamma(\alpha)}\int_0^1 (1-v)^{\alpha-1} f(xv)\,dv,$$

and the integral of the right-hand side is infinite if f is non-negative and not identically zero. In this respect, let us also notice that the arbitrary constant $\Gamma(\alpha)$ in (1) was chosen without loss of generality: if $f_{m,\alpha}$ is a density solution to (1), then $f_{c,m,\alpha}(x) = cf_{m,\alpha}(cx)$ is for every $c > 0$ a density solution to

$$x^m f(x) \;=\; \frac{c^{\alpha-m}}{\Gamma(\alpha)}\int_0^x (x-v)^{\alpha-1} f(v)dv.$$

The study of density solutions to (1) for m a positive integer was initiated in [29] and then pursued in [27], where the corresponding random variables are called “generalized stable”. Apart from the classical stable case $m = 1$, these random variables are of interest in the case $\{m = 2, \alpha \in (0,1)\}$ which is especially investigated in Section 3 of [29] and Section 7 of [27], because of its connections to particle transport along the one-dimensional lattice – see [2]. The paper [29] takes the point of view of Fox functions and shows that for all $m \in \mathbb{N}^*, \alpha \in (0,1)$ there exists a density solution to (1) having a convergent power series representation at infinity and a Fréchet-like behaviour at zero – see (2.12) and (2.15) therein. The paper [27] takes the point of view of size-biasing and shows that for all $m \in \mathbb{N}^*, \alpha \in (0,m)$ there exists a unique density solution to (1), whose corresponding random variable can be represented in the case $\alpha \in (m-1,m)$ as a finite independent product involving the Gamma and positive stable random variables – see Theorems 4.3 and 4.2 therein.

Before stating our main result, let us consider a few simple and explicit examples, related to the Gamma and positive stable random variables as in [27], but in a more elementary way. When $\alpha = n$ is a positive integer, then (1) becomes an ODE of order n satisfied by the n-th cumulative distribution function

$$F_n(x) \;=\; \int_{0<x_1<\ldots<x_n<x} f(x_1)\,dx_1\ldots dx_n,$$

which is

$$F_n \;=\; x^m F_n^{(n)}.$$

- For $\alpha = 1$, we solve $F_1 = x^m F_1'$ with F_1 bounded and vanishing at zero. This implies that F_1' is a density iff $m > 1$ with $F_1(x) = e^{-\frac{x^{1-m}}{(m-1)}}$, that is $f_{m,1} = F_1'$ is the density of the Fréchet random variable $((m-1)\mathbf{\Gamma}_1)^{\frac{1}{1-m}}$ where, here and throughout, $\mathbf{\Gamma}_t$ denotes a Gamma random variable of parameter $t > 0$, with density