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Elliptic differential operators and positive semigroups associated with generalized Kantorovich operators ☆

Francesco Altomare [a,*], Mirella Cappelletti Montano [a], Vita Leonessa [b], Ioan Raşa [c]

[a] *Dipartimento di Matematica, Università degli Studi di Bari "A. Moro", Campus Universitario, Via E. Orabona n. 4, 70125-Bari, Italy*
[b] *Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, Viale Dell' Ateneo Lucano n. 10, Campus di Macchia Romana, 85100-Potenza, Italy*
[c] *Department of Mathematics, Technical University of Cluj-Napoca, Str. Memorandumului 28, RO-400114 Cluj-Napoca, Romania*

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A B S T R A C T

Deepening the study of a new approximation sequence of positive linear operators we introduced and studied in [12], in this paper we disclose its relationship with the Markov semigroup (pre)generation problem for a class of degenerate second-order elliptic differential operators which naturally arise through an asymptotic formula, as well as with the approximation of the relevant Markov semigroups in terms of the approximating operators themselves.
The analysis is carried out in the context of the space $\mathscr{C}(K)$ of all continuous functions defined on an arbitrary convex compact subset K of $\mathbf{R}^d$, $d \geq 1$, having non-empty interior and a not necessarily smooth boundary, as well as, in some particular cases, in $L^p(K)$ spaces, $1 \leq p < +\infty$. The approximation formula also allows to infer some preservation properties of the semigroup such as the preservation of the Lipschitz-continuity as well as of the convexity. We finally apply the main results to some noteworthy particular settings such as balls and ellipsoids, the unit interval and multidimensional hypercubes and simplices. In these settings the relevant differential operators fall into the class of Fleming–Viot operators.

0. Introduction

In [12] we introduced and studied a new sequence $(C_n)_{n\geq 1}$ of positive linear operators acting on function spaces defined on a convex compact subset K of some locally convex Hausdorff space. Their construction depends on a given Markov operator $T : \mathscr{C}(K) \to \mathscr{C}(K)$, a real number $a \geq 0$ and a sequence $(\mu_n)_{n\geq 1}$ of probability Borel measures on K.

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* Corresponding author.
E-mail addresses: francesco.altomare@uniba.it (F. Altomare), mirella.cappellettimontano@uniba.it (M. Cappelletti Montano), vita.leonessa@unibas.it (V. Leonessa), Ioan.Rasa@math.utcluj.ro (I. Raşa).

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More precisely, for every $n \geq 1$, they are defined by setting

$$C_n(f)(x) = \int_K \cdots \int_K f\left(\frac{x_1 + \ldots + x_n + ax_{n+1}}{n+a}\right) d\tilde{\mu}_x^T(x_1) \cdots d\tilde{\mu}_x^T(x_n) d\mu_n(x_{n+1})$$

for every $x \in K$ and for every $f \in \mathscr{C}(K)$, where $(\tilde{\mu}_x^T)_{x \in K}$ is the continuous selection of probability Borel measures on K corresponding to T via the Riesz representation theorem.

For particular choices of these parameters and for particular convex compact subsets, such as the unit interval or the multidimensional hypercube and simplex, these operators turn into the Kantorovich operators and in several of their wide-ranging generalizations.

In [12] we mainly investigated and studied the approximation properties of these operators in the space $\mathscr{C}(K)$ and, in some cases, in L^p-spaces, $1 \leq p < +\infty$.

Under the influence of a series of researches developed during the last two decades, which are concerned with the relationship between degenerate differential operators, Markov semigroups and approximation processes (see, e.g., [4] and [11]), it has been quite natural to investigate, in the special case when $K \subset \mathbf{R}^d$, $d \geq 1$, whether, by using the theory of one-parameter semigroups, the new approximation process can also be used to solve some classes of initial-boundary value problems associated with suitable degenerate differential operators as well as to approximate the solutions of such differential problems.

From an operator theoretical point of view this problem corresponds to determine the differential operator generated by an asymptotic formula for the approximating operators C_n and to investigate whether it (pre)generates a (positive) C_0-semigroup which, in turn, can be approximated in terms of suitable iterates of them.

In the present setting, assuming that the sequence $(\mu_n)_{n \geq 1}$ is weakly convergent to some (probability) Borel measure μ on K, then the differential operators which arise through such a method are of the form

$$V(u)(x) = \frac{1}{2} \sum_{i,j=1}^{d} \alpha_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} a(b_i - x_i) \frac{\partial u}{\partial x_i}(x)$$

$(u \in \mathscr{C}^2(K), x = (x_1, \ldots, x_d) \in K)$, where

$$\alpha_{ij} := T(pr_i pr_j) - pr_i pr_j \qquad (i, j = 1, \ldots, d),$$

each pr_i denoting the i-th coordinate function $(i = 1, \ldots, d)$, and $b = (b_1, \ldots, b_d) \in K$ stands for the barycenter of the measure μ. Moreover, the coefficients α_{ij} vanish on a subset of the boundary of K which contains the extreme points of K.

Under the assumptions that T leaves invariant the continuous affine functions on K and maps polynomials into polynomials of at most the same degree, we show, indeed, that $(V, \mathscr{C}^2(K))$ is the pregenerator of a Markov semigroup on $\mathscr{C}(K)$ which is approximated in terms of suitable iterates of the C_n's. This semigroup is referred to as the limit semigroup of the C_n's.

Specializing the convex compact set K and the other parameters, we obtain several classes of differential operators which are of current interest in the research area of evolution equations. Among them we quote the degenerate diffusion operators on balls and ellipsoids ([11], [28]) and the Fleming–Viot type operators on the unit interval and on the multidimensional hypercube and simplex ([2], [3], [5], [7], [11], [15], [21], [24]).

Our approach allows to study all these particular cases in an unifying manner and also to obtain some extensions of the existing generation results.

However, the main feature of the paper rests not only on the study of the generation results for the differential operators as above in the framework of convex compact domains with not necessarily smooth