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Commutator estimates for the Dirichlet-to-Neumann map of Stokes systems in Lipschitz domains

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A B S T R A C T

In the paper, we establish commutator estimates for the Dirichlet-to-Neumann map of Stokes systems in Lipschitz domains. The approach is based on Dahlberg's bilinear estimates, and the results may be regarded as an extension of [8,21] to Stokes systems.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain with $d \geq 3$. It is well known that for any $f \in H^{1/2}(\partial\Omega;\mathbb{R}^d)$ with the compatibility condition $\int_{\partial\Omega} n \cdot f dS = 0$, the Dirichlet problem for the Stokes system

$$\begin{cases} \Delta u - \nabla q = 0 & \text{in} \quad \Omega, \\ \operatorname{div}(u) = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

has a unique velocity u in $V_N = \{v \in H^1(\Omega;\mathbb{R}^d) : \operatorname{div}(v) = 0\}$, and a unique pressure q up to constants in $L^2(\Omega)$. To make the following definition well-defined, we may assume $\int_\Omega q(x)dx = 0$. The Dirichlet-to-Neumann map $\Lambda : H^{1/2}(\partial\Omega;\mathbb{R}^d) \to H^{-1/2}(\partial\Omega;\mathbb{R}^d)$ is defined by

$$\big(\Lambda(f)\big)^\alpha = \frac{\partial u^\alpha}{\partial n} - n_\alpha q \tag{1.2}$$

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in a weak sense, where $n = (n_1, \cdots, n_d)$ is the outward unit normal to $\partial\Omega$. The right-hand side of (1.2) denotes the conormal derivative of u on $\partial\Omega$ (see for example [11,18]). Furthermore, from the results in [11], one may show that $\|\Lambda(f)\|_{L^2(\partial\Omega)} \leq C\|f\|_{H^1(\partial\Omega)}$.

In the paper, we will study the L^2-theory of the commutator estimates for the Dirichlet-to-Neumann map (1.2), and the main results will be shown in the following.

Theorem 1.1. *Let Ω be a bounded Lipschitz domain, and $f \in L^2(\partial\Omega; \mathbb{R}^d)$ satisfy the compatibility condition $\int_{\partial\Omega} n \cdot f dS = 0$. Suppose (u, q) is the solution of (1.1) with boundary data f. Then for any $\eta \in C^{0,1}(\partial\Omega)$ satisfying $\int_{\partial\Omega} n \cdot \eta f dS = 0$, we have*

$$\big\|\Lambda(\eta f) - \eta\Lambda(f)\big\|_{L^2(\partial\Omega)} \leq C\|\eta\|_{C^{0,1}(\partial\Omega)}\|f\|_{L^2(\partial\Omega)}, \tag{1.3}$$

where C depends on d and Ω. Particularly, in the case of $d = 3$, the estimate

$$\big\|\Lambda(\eta f) - \eta\Lambda(f)\big\|_{L^2(\partial\Omega)} \leq C\|\eta\|_{H^1(\partial\Omega)}\|f\|_{L^\infty(\partial\Omega)}, \tag{1.4}$$

also holds, where C depends only on Ω.

Remark 1.2. In the proof of (1.4), the assumption that $d = 3$ merely guarantees the L^∞-estimate $\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\partial\Omega)}$ is valid, which is well known as the Agmon–Miranda maximum principle in the field of elliptic systems. Whether such the L^∞-estimate holds in Lipschitz domains for $d \geq 4$ remains an interesting open problem.

The estimates (1.3) and (1.4) are referred to as the commutator estimates. The key step in the proof of Theorem 1.1 is to establish the following Dahlberg's bilinear estimate

$$\begin{aligned}\Big|\int_\Omega \nabla u \cdot v dx\Big| \leq C\Big\{\Big(\int_\Omega |\nabla u|^2\delta(x)dx\Big)^{\frac{1}{2}} + \Big(\int_\Omega |q|^2\delta(x)dx\Big)^{\frac{1}{2}}\Big\} \\ \times \Big\{\Big(\int_\Omega |\nabla v|^2\delta(x)dx\Big)^{\frac{1}{2}} + \Big(\int_{\partial\Omega} |(v)^*|^2 dS\Big)^{\frac{1}{2}}\Big\}\end{aligned} \tag{1.5}$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$, and $v = (v_i^\alpha) \in H^1(\Omega; \mathbb{R}^{d\times d})$. The notation $(v)^*$ in (1.5) represents the nontangential maximal function of v on $\partial\Omega$, defined by

$$(v)^*(x) = \sup_{y\in\Gamma_{N_0}(x)} |v(y)|, \qquad \Gamma_{N_0}(x) = \big\{y \in \Omega : |y - x| \leq N_0\mathrm{dist}(y, \partial\Omega)\big\},$$

where $x \in \partial\Omega$, and N_0 is sufficiently large. The bilinear estimate was originally proved in [7] for harmonic functions in Lipschitz domains. In term of the elliptic system with variable coefficients, it was established by S. Hofmann [13], and by Z. Shen [21], respectively, for different considerations. In fact, this work is much influenced by [21].

Compared to the bilinear estimate established for elliptic equations (see [7,8,21,13]), the estimate (1.5) has one more square function caused by the pressure term q, and how to handle that term will be the main difficulty in the technical standpoint. In term of layer potential, we have the key observation that $\Delta q = 0$ in $\mathbb{R}^d \setminus \partial\Omega$, which leads two important facts. One is that the square function of q may be controlled by the boundary data (see Lemma 2.1), which is based on the equivalence between the square function and the nontangential maximal function (see [2,10]). The other is that $|q(x)|^2\delta(x)dx$ could be a Carleson measure provided the velocity term u is bounded. Although these results may probably be known by experts, a rigorous proof seems to have considerable merit, and benefits the readers.

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