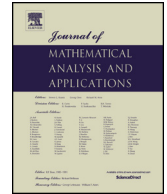




Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa



# Almost sure exponential stability of hybrid stochastic functional differential equations

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## ARTICLE INFO

### Article history:

Received 25 June 2017

Available online xxxx

Submitted by U. Stadtmueller

### Keywords:

Stability

Hybrid stochastic differential functional equations

Itô formula

Brownian motion

Markov chain

## ABSTRACT

This paper is concerned with the almost sure exponential stability of the  $n$ -dimensional nonlinear hybrid stochastic functional differential equation (SFDE)  $dx(t) = f(\psi_1(x_t, t), r(t), t)dt + g(\psi_2(x_t, t), r(t), t)dB(t)$ , where  $x_t = \{x(t+u) : -\tau \leq u \leq 0\}$  is a  $C([-\tau, 0]; \mathbb{R}^n)$ -valued process,  $B(t)$  is an  $m$ -dimensional Brownian motion while  $r(t)$  is a Markov chain. We show that if the corresponding hybrid stochastic differential equation (SDE)  $dy(t) = f(y(t), r(t), t)dt + g(y(t), r(t), t)dB(t)$  is almost surely exponentially stable, then there exists a positive number  $\tau^*$  such that the SFDE is also almost surely exponentially stable as long as  $\tau < \tau^*$ . We also describe a method to determine  $\tau^*$  which can be computed numerically in practice.

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## 1. Introduction

This paper is concerned with the almost sure exponential stability of the  $n$ -dimensional nonlinear hybrid stochastic functional differential equation (SFDE) of the form

$$dx(t) = f(\psi_1(x_t, t), r(t), t)dt + g(\psi_2(x_t, t), r(t), t)dB(t). \quad (1.1)$$

Here  $B(t)$  is an  $m$ -dimensional Brownian motion,  $r(t)$  is a Markov chain on the finite state space  $\mathbb{S} = \{1, 2, \dots, N\}$ ,  $x_t = \{x(t+s) : -\tau \leq s \leq 0\}$ ,  $\tau$  is a positive number,  $\psi_1, \psi_2 : C([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ ,  $f : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times m}$ . The notation used will be explained in Section 2 while we refer the reader to, for example, [9–12, 19, 20] for the general theory on SFDEs.

To see the difficulty of this problem, let us recall some history in the area of almost sure stability of SFDEs. In 1997, Mohammed and Scheutzow [21] were first to study the almost sure exponential stability of the linear scalar stochastic differential delay equation (SDDE, a special class of SFDEs)

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$$dx(t) = \sigma x(t - \tau)dB(t), \tag{1.2}$$

where  $B(t)$  is a scalar Brownian motion and  $\sigma$  is positive number. They showed that the SDDE (1.2) is almost surely exponentially stable provided the time delay  $\tau$  is sufficiently small. Their proof for this was nontrivial. In 2005, Scheutzow [23] considered a more general scalar SFDE

$$dx(t) = \sigma\psi(x_t)dB(t), \tag{1.3}$$

where  $\sigma$  is positive number and  $\psi$  is a Lipschitz continuous functional from  $C([-\tau, 0]; \mathbb{R})$  to  $\mathbb{R}$  such that

$$\inf_{-\tau \leq s \leq 0} |\varphi(s)| \leq |\psi(\varphi)| \leq \sup_{-\tau \leq s \leq 0} |\varphi(s)|, \quad \forall \varphi \in C([-\tau, 0]; \mathbb{R}).$$

He also showed that equation (1.3) is almost surely exponentially stable provided  $\tau$  is sufficiently small. In 2016, Guo et al. [7] considered the more general  $n$ -dimensional nonlinear SDDE with variable delays of the form

$$dx(t) = f(x(t - \delta_1(t)), t)dt + g(x(t - \delta_2(t)), t)dB(t), \tag{1.4}$$

where  $B(t)$  is an  $m$ -dimensional Brownian motion,  $\delta_1, \delta_2 : \mathbb{R}_+ \rightarrow [0, \tau]$  stand for variable delays, while  $f : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times m}$  are globally Lipschitz continuous. They showed that if the corresponding (non-delay) SDE

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \tag{1.5}$$

is almost surely exponentially stable, so is the SDDE (1.4) provided the time delays are sufficiently small. The reason why it has taken almost 20 years to make these progresses in this area is because SFDEs (including SDDEs) are infinite-dimensional systems which are significantly different from SDEs. For example, it is straightforward to show that the linear scalar SDE  $dx(t) = \sigma x(t)dB(t)$  is almost surely exponentially stable by applying the Itô formula to  $\log(x(t))$  (see, e.g. [2,6]). However, it is nontrivial for Mohammed and Scheutzow [21] to show the almost sure exponential stability of the corresponding SDDE (1.2) for sufficiently small  $\tau$  and they used a different approach (as one cannot apply the Itô formula to  $\log(x(t))$  in this delay case).

The underlying SFDE (1.1) in this paper is more general than any of equations (1.2), (1.3) or (1.4). This is not only because of the hybrid factor modelled by the Markov chain  $r(t)$  but also more general without the Markov chain. In fact, ignoring  $r(t)$  and setting  $\psi_1(x_t, t) = x(t - \delta_1(t))$  and  $\psi_2(x_t, t) = x(t - \delta_2(t))$ , we see that the SFDE (1.1) becomes equation (1.4); while if we set  $f = 0, g(x, i, t) = \sigma x$  and  $\psi_2(x_t, t) = \psi(x_t)$ , then the SFDE (1.1) becomes equation (1.3).

All of the above show the difficulty and generality of our proposed problem. Let us begin to develop our new theory.

## 2. Preliminaries

Throughout this paper, unless otherwise specified, we will use the following notation. Let  $|x|$  denote the Euclidean norm of vector  $x \in \mathbb{R}^n$ . For a matrix  $A$ , let  $|A| = \sqrt{\text{trace}(A^T A)}$  be its trace norm and  $\|A\| = \max\{|Ax| : |x| = 1\}$  be the operator norm. For a vector or matrix  $A$ , its transpose is denoted by  $A^T$ . If  $A$  is a symmetric real matrix ( $A = A^T$ ), denote by  $\lambda_{\min}(A)$  and  $\lambda_{\max}(A)$  its smallest and largest eigenvalue, respectively.

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  be a complete probability space with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the usual conditions. Let  $B(t) = (B_1(t), \dots, B_m(t))^T$  be an  $m$ -dimensional Brownian motion with respect to the

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