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Subprojective Nakano spaces

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ABSTRACT

A Banach space X is subprojective if every infinite-dimensional subspace of X has a subspace which is complemented in X. We prove that separable Nakano sequence spaces $\ell_{(p_n)}$ are subprojective. Subprojectivity is also characterized in separable Nakano function spaces $L^{p(\cdot)}(0,1)$ and $L^{p(\cdot)}(0,\infty)$.

1. Introduction

A Banach space X is subprojective if every infinite-dimensional subspace of X has a infinite-dimensional subspace which is complemented in X. Several examples of subprojective spaces can be found in [14]. For instance, the spaces ℓ_p $(1 \leq p < \infty)$ and c_0 are subprojective. However, the space ℓ_∞ is not subprojective since it is a prime space (see [8, Theorem 2.a.7]) and every separable Banach space embeds isometrically into ℓ_∞ (see [1, Theorem 2.5.7]). On the other side, the space $L^p(0,1)$ is subprojective if and only if $2 \leq p < \infty$.

Recently, Oikhberg and Spinu have studied in [12] the stability of subprojectivity of Banach spaces under various operations, such as direct or twisted sums, tensor products, and forming spaces of operators, obtaining new classes of subprojective spaces. This paper is a further contribution to the study of subprojectivity.

More examples related to operator theory can be found in [3,4].

Given a Banach space, it is necessary to study its own complemented structure in order to solve the subprojectivity problem. Orlicz sequence spaces are very close to ℓ_p-spaces (both spaces have symmetric basis) and those can be non-subprojective (see [8, Example 4.c.3]).

Nakano function spaces (or variable exponent Lebesgue spaces) $L^{p(\cdot)}(\Omega)$ belong to the class of Musielak–Orlicz spaces (see [10]). A strong interest in these spaces has appeared in the last decade (see e.g. [2]).

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Nakano function spaces are close to L^p-spaces but they do not have symmetric structure. Furthermore, the averaging projection is not bounded on Nakano function spaces (see [7, Theorem 4.4]). The geometric properties of Nakano function spaces can be found in [6] (see [13] for Nakano sequence spaces $\ell_{(p_n)}$).

In section 3 we study the complemented structure of separable Nakano sequence spaces, showing how projections which work for ℓ_p-spaces fail for $\ell_{(p_n)}$-spaces. In section 4 we adapt the results in section 3 to the case of separable Nakano function spaces.

In section 5 we prove that separable Nakano sequence spaces are subprojective. If $\ell_{(p_n)}$ is not separable, then ℓ_∞ embeds isomorphically into $\ell_{(p_n)}$ (see [6, Proposition 3.1]) and the space is not subprojective. Moreover, using the results in [6] on subspaces of separable Nakano function spaces, we prove that $L^{p(\cdot)}(0,1)$ is subprojective if and only if it does not contain a subspace isomorphic to ℓ_q with $q < 2$.

Throughout this paper, all subspaces are assumed to be closed, until specified otherwise.

2. Preliminaries

We will use the standard Banach space terminology (see [1,8,9]).

We will denote by $[d_k]$ to the closed subspace spanned by the basic sequence $(d_k)_k$.

The basic sequences $(d_k)_k$ and $(e_k)_k$ are said to be *equivalent* whenever the series $\sum_{k=1}^\infty x_k d_k$ converges if and only if the series $\sum_{k=1}^\infty x_k e_k$ converges for every scalar sequence $(x_k)_k$. We will write $[d_k] \simeq [e_k]$.

We say that two basic sequences $(d_k)_k$ and $(e_k)_k$ in a Banach space X are *BP-equivalent* (equivalent in the sense of Bessaga–Pełzyński) whenever

$$\sum_{k=1}^\infty \|d_k - e_k\|_X < \infty.$$

In this paper we will consider only monotone basic sequences and norm one projections. Therefore, by [8, Proposition 1.a.9], we have that BP-equivalent basis are equivalent basis and complementedness is preserved.

Let (Ω, Σ, μ) be a complete σ-finite separable non-atomic measurable space and let us consider $L_0(\Omega)$, the space of all real measurable function classes on Ω. If $p : \Omega \longrightarrow [1,\infty)$ is a measurable function, the *Nakano function space* $L^{p(\cdot)}(\Omega)$ (or variable exponent Lebesgue space) is the set of all $f \in L_0(\Omega)$ such that

$$\rho_{p(\cdot)}(f/r) = \int_\Omega \left(\frac{|f(t)|}{r}\right)^{p(t)} d\mu(t) < \infty$$

for some $r > 0$, endowed with the Luxemburg norm

$$\|f\|_{p(\cdot)} = \inf\{r > 0 : \rho_{p(\cdot)}(f/r) \le 1\}.$$

We write $p^+ = \text{ess sup}\{p(t) : t \in \Omega\}$ and $p^- = \text{ess inf}\{p(t) : t \in \Omega\}$. By $p|_B^+$ and $p|_B^-$ we denote the essential supremum and infimum of the function p over a measurable subset B. It holds that $p^+ < \infty$ if and only if $L^{p(\cdot)}(\Omega)$ is separable, and $1 < p^- \le p^+ < \infty$ if and only if $L^{p(\cdot)}(\Omega)$ is reflexive. The topological dual of $L^{p(\cdot)}(\Omega)$ is $L^{q(\cdot)}(\Omega)$ where the function $q(\cdot)$ is defined by the equation $\frac{1}{p(t)} + \frac{1}{q(t)} = 1$ almost everywhere $t \in \Omega$ (see [2]).

Given two real number sequences $(p_n)_n \subset [1,\infty)$ and $(w_n)_n \subset (0,\infty)$, the *weighted Nakano sequence space* $\ell_{(p_n)}(w_n)$ is the Banach space

$$\ell_{(p_n)}(w_n) = \left\{(x_n)_n \in \mathbb{R}^{\mathbb{N}} : \rho((x_n)_n) = \sum_{n=1}^\infty \left|\frac{x_n}{r}\right|^{p_n} w_n < \infty \text{ for some } r > 0\right\}$$