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On the duality between rotational minimal surfaces and maximal surfaces

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ABSTRACT

We investigate the duality between minimal surfaces in Euclidean space and maximal surfaces in Lorentz–Minkowski space in the framework of rotational surfaces. We study if the dual surfaces of two congruent rotational minimal (or maximal) surfaces are congruent. We analyze the duality process when we deform a rotational minimal (maximal) surface by a one-parametric group of rotations. In this context, the family of Bonnet minimal (maximal) surfaces and the Goursat transformations play a remarkable role.

1. Introduction

There is a correspondence, known as duality, between minimal surfaces in Euclidean space $\mathbf{E}^3$ and maximal surfaces in Lorentz–Minkowski space $\mathbf{L}^3$. This correspondence assigns a maximal surface to each minimal surface and *vice-versa* and it was introduced by Calabi for minimal surfaces and maximal surfaces when are expressed as graphs on a simply-connected domain [5]. A similar correspondence between both families of surfaces also appeared on [8,14], where the duality is now defined in terms of the isotropic curve that determines the surface and finally, H. Lee proved in [11] that both methods are equivalent. We describe this correspondence. If $X:\Omega\to\mathbf{E}^3$ is a conformal minimal surface defined on a simply-connected domain Ω of the complex plane $\mathbb{C}$ and z is the conformal parameter, then the complex curve $\phi:\Omega\to\mathbb{C}^3$ defined by $\phi(z)=2X_z=(\phi_1,\phi_2,\phi_3)$ is holomorphic and satisfies the isotropy relation $\langle\phi,\phi\rangle=\phi_1^2+\phi_2^2+\phi_3^2=0$. If we now define $\psi:\Omega\to\mathbb{C}^3$ by $(\psi_1,\psi_2,\psi_3)=(-i\phi_1,-i\phi_2,\phi_3)$, then $\psi_1^2+\psi_2^2-\psi_3^2=0$ and consequently ψ determines a maximal surface $X^\flat:\Omega\to\mathbf{L}^3$ by setting $X^\flat(z)=\Re\int^z\psi(z)dz$, where the integral does not depend on the path because Ω is simply-connected. This process has its converse: if $X:\Omega\to\mathbf{L}^3$ is a conformal maximal surface and $\psi(z)=2X_z=(\psi_1,\psi_2,\psi_3)$, then the complex curve $\phi=(i\psi_1,i\psi_2,\psi_3)$

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satisfies $\langle\phi,\phi\rangle = 0$ and defines a minimal surface $M^\sharp$ in $\mathbf{E}^3$ by means of $X^\sharp(z) = \Re\int^z \phi(z)dz$. Furthermore, and up to translations of the ambient space, we have $M = (M^\sharp)^\flat$. If Min and Max denote the family of minimal surfaces of $\mathbf{E}^3$ and the maximal surfaces of $\mathbf{L}^3$, respectively, we have established two maps

$$\flat : \text{Min} \to \text{Max}, \qquad \sharp : \text{Max} \to \text{Min}$$

with the property that $\flat \circ \sharp$ and $\sharp \circ \flat$ are the identities in Max and Min respectively. We say that $M^\flat$ (or $M^\sharp$) is the *dual surface* of M (also named in the literature as the twin surface [8,12,15]). This process of duality has been generalized in other ambient spaces: see for example, [1,12,19].

In this paper we are interested in a problem posed by Araujo and Leite in [3] that asks whether the dual surfaces of two congruent minimal (or maximal) surfaces are also congruent. We precise the terminology. We say that two surfaces M_1 and M_2 of $\mathbf{E}^3$ (or $\mathbf{L}^3$) are congruent, and we denote by $M_1 \simeq M_2$, if there is an orientation-preserving isometry of the ambient space carrying M_1 onto M_2. Here we also suppose that this relation $\simeq$ is up to an automorphism on M_1 and M_2 and up to dilations of the ambient space because dilations preserve the zero mean curvature property. In $\mathbf{E}^3$ the set of congruences preserving the orientation is $SO(3)$ and in $\mathbf{L}^3$ is $SO(2,1)$. Then the problem can be formulated as follows:

Problem 1. If $M_1 \simeq M_2$ are two minimal surfaces of $\mathbf{E}^3$, does $M_1^\flat \simeq M_2^\flat$ hold?

A similar question can be posed for maximal surfaces. Surprisingly the answer is 'not in general' and there are many congruent minimal surfaces whose dual surfaces are not congruent. In other words, if $\{T(M) : T \in SO(3)\}$ is the congruent class of a minimal surface $M \subset \mathbf{E}^3$, the quotient space $\{T(M)^\flat : T \in SO(3)\}/\simeq$ has many elements. In [3] the authors study this problem in the case that M is the Enneper surface, the Scherk surface and the catenoid.

In this paper we focus how the geometric properties of a minimal (or maximal) surface can transform to its dual surface and we pay our attention for rotational surfaces with the next question:

Problem 2. Is the dual surface rotational of a rotational minimal (or maximal) surface?

Let us observe that the maps $\flat$ and $\sharp$ do not carry any geometrical information of the surface because in the definition of a dual surface only it is involved the complex coordinates of the surface. We point out that the class of rotational surfaces of $\mathbf{L}^3$ is richer than in the Euclidean case because in $\mathbf{L}^3$ there exist three types of rotational surfaces according to the causal character of the axis of revolution. If we restrict to maximal surfaces, there are three types of non-congruent rotational maximal surfaces, named, elliptic catenoid, hyperbolic catenoid and parabolic catenoid when the rotational axis is timelike, spacelike or lightlike, respectively [10]. It was proved in [3] that for an Euclidean catenoid M, the quotient space $\{T(M)^\flat : T \in SO(3)\}/\simeq$ has the topology of the closed interval $[0,1]$, obtaining a Bonnet maximal surface for $t < 1$ and the hyperbolic catenoid for $t = 1$. In Section 4 we recover this result by describing explicitly the dual surfaces of the rotational maximal surfaces of $\mathbf{L}^3$. Then we take an Euclidean catenoid C with axis $(0,0,1)$ and we consider the dual surfaces of the Euclidean catenoids obtained by rotating C about a line orthogonal to the axis of C obtaining many non-congruent maximal surfaces. In Section 5 we consider each one of the three catenoids C of $\mathbf{L}^3$ and we deform it by a one-parametric group of rotations about an axis which will have a different causal character than the one of C. We will prove in Theorems 5.2, 5.3 and 5.4 that the dual surfaces belong to the Bonnet family of minimal surfaces up to a Goursat transformation or it is the Enneper surface.

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