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Testing families of complex lines for the unit ball

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ABSTRACT

Let P_1 and P_2 be two points in $\mathbb{C}^2$ such that the complex line joining them is tangent to the unit sphere $\partial\mathbb{B}^2$. We prove that if a real analytic function f on $\partial\mathbb{B}^2$ extends holomorphically on every complex line through P_1 and P_2, then f extends holomorphically to the whole ball $\mathbb{B}^2$.

Contents

1. Introduction

Let $f : \partial\mathbb{B}^2 \to \mathbb{C}$ be a continuous function and let l be a complex line in $\mathbb{C}^2$. Suppose that l and $\mathbb{B}^2$ have non-empty intersection and let $D = l \cap \mathbb{B}^2$ be the disc resulting from such intersection. We will say that f extends holomorphically to l (or to D) if there exists a function $\tilde{f} : \overline{D} \to \mathbb{C}$ holomorphic in D and continuous up to the boundary of D such that $f|_{\partial D} = \tilde{f}|_{\partial D}$. We will often refer to D as the complex line l even though it is just a piece of l. If l does not intersect $\mathbb{B}^2$ we will also say that f extends holomorphically to l (in this case there is no actual requirement on f). The problem of describing families of discs which suffice for testing analytic extension of a function f from the sphere $\partial\mathbb{B}^2$ to the ball $\mathbb{B}^2$ has a long history. For f continuous on $\partial\mathbb{B}^2$, Agranovsky–Valskii [4] use all the lines, Agranovsky–Semenov [3] the lines through an open subset of $\mathbb{B}^2$, Rudin [12] the lines tangent to a concentric subsphere, Baracco–Tumanov–Zampieri [8]

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the lines tangent to any strictly convex subset of $\mathbb{B}^2$. There are several other contributions in this direction, such as [1] and [11]. It is a challenging attempt to reduce the number of parameters in the testing families. However, one encounters an immediate constraint: lines which meet a single point $z_o \in \mathbb{B}^2$ do not suffice. Instead, two interior points or a single boundary point suffice: Agranovsky [2] and Baracco [5–7]. However, in these last two results, the reduction of the testing families is compensated by additional initial regularity of f: this is assumed to be real analytic. Globevnik [9,10] shows that, for two points whose joining line meets the interior of the ball, C^∞-regularity still suffices, but C^k does not. This suggests that holomorphic extension is a good balance between reduction of testing families and improvement of initial regularity. Note that if the joining line misses the closed ball $\overline{\mathbb{B}^2}$ then two points do not suffice, not even for real analytic functions [10]. In this paper, we investigate the case of two points whose joning line is tangent to the ball. We show that in this case the set of lines passing through these points and having non-empty intersection with $\mathbb{B}^2$ suffice for testing holomorphicity for real analytic functions. Our main result is the following.

Theorem 1.1. *Let P_1, P_2 be two points in $\mathbb{C}^2 \setminus \overline{\mathbb{B}^2}$ such that the complex line joining them is tangent to the sphere $\partial\mathbb{B}^2$. If a real analytic function $f \in C^\omega(\partial\mathbb{B}^2)$ extends holomorphically on every line through P_1 and every line through P_2, then f extends holomorphically to $\mathbb{B}^2$.*

Together with [9,10] Theorem 1.1 yields the following

Corollary 1.2. *Let P_1 and P_2 be two points in $\mathbb{C}^2$ such that the complex line joining them has non-empty intersection with the closed ball $\overline{\mathbb{B}^2}$. If a function $f \in C^\omega(\partial\mathbb{B}^2)$ extends holomorphically on every line through P_1 and every line through P_2, then f extends holomorphically to $\mathbb{B}^2$.*

2. Proof of the main theorem

After a rotation we can assume that P_1 and P_2 belong to the complex line of equation $z_2 = 1$, and that they are different from the point $N = (0, 1)$. By composing with an automorphism of the ball which fixes N (see (2.12)), we can assume that P_1 is the point at infinity of the line $z_2 = 1$. With this choice, the lines through P_1 are the lines parallel to the $(1, 0)$-direction. We first extend f in a real analytic fashion in a neighborhood of N and call this extension again f. In general, we have

$$f(z_1, z_2, \overline{z}_1, \overline{z}_2) = \sum_{\alpha,\beta,\gamma,\delta\in\mathbb{N}} a_{\alpha\beta\gamma\delta} z_1^\alpha \overline{z}_1{}^\beta (z_2 - 1)^\gamma (\overline{z}_2 - 1)^\delta. \tag{2.1}$$

We remark that we can choose such an extension near N so that some of the $a_{\alpha\beta\gamma\delta}$ are 0. More precisely, since $\bar{\partial}_{z_2}$ is non-characteristic for $\partial\mathbb{B}^2$ at N, by the Cauchy–Kowalevskaya Theorem we can choose an extension which is holomorphic in z_2. Near N we thus have

$$f(z_1, \bar{z_1}, z_2) = \sum_{\alpha,\beta,\gamma\in\mathbb{N}} a_{\alpha\beta\gamma} z_1^\alpha \overline{z}_1^\beta (z_2 - 1)^\gamma. \tag{2.2}$$

With this setup, the theorem is proved once we show that $a_{\alpha\beta\gamma} = 0$ for $\beta > 0$. In fact, if $a_{\alpha\beta\gamma} = 0$ for $\beta > 0$, then f is holomorphic in a neighborhood U of N. In particular, f is CR on $U \cap \partial\mathbb{B}^2$. This means that $Xf = 0$ on $U \cap \partial\mathbb{B}^2$, where $X = z_1\overline{\partial}_{z_2} - z_2\overline{\partial}_{z_1}$ is the CR vector field of the sphere. By real analiticity we have that $Xf = 0$ on the whole sphere, hence $f \in CR(\partial\mathbb{B}^2)$, and it extends holomorphically to the ball by the classical Bochner–Hartogs extension theorem.

2.1. First part of the proof: $a_{\alpha\beta\gamma} = 0$ for $\alpha < \beta$

We need the following lemma.