J. Math. Anal. Appl. 458 (2018) 419–435

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Invariants for the Lagrangian equivalence problem

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A R T I C L E I N F O

*Article history:*
Received 2 May 2017
Available online 5 September 2017
Submitted by G. Bluman

*Keywords:*
Fibred manifold
Jet bundle
Lagrangian density
Lagrangian equivalence problem

A B S T R A C T

Let $M$ be a connected smooth manifold, let $\mathrm{Aut}(p)$ be the group automorphisms of the bundle $p\colon \mathbb{R}\times M\to\mathbb{R}$, and let $q\colon J^1(\mathbb{R},M)\times\mathbb{R}\to J^1(\mathbb{R},M)$ be the canonical projection. Invariant functions on $J^r(q)$ under the natural action of $\mathrm{Aut}(p)$ are discussed in relationship with the Lagrangian equivalence problem. The second-order invariants are determined geometrically as well as some other higher-order invariants for $\dim M\geq 2$.



## 1. Introduction and preliminaries

### *1.1. Statement of the problem*

Let $M$ be a connected smooth manifold of dimension $m$. An automorphism of the projection $p\colon \mathbb{R}\times M\to\mathbb{R}$, $p(x,y)=x$, is a pair $\phi\in\mathrm{Diff}\,\mathbb{R}$, $\Phi\in\mathrm{Diff}(\mathbb{R}\times M)$ making the following diagram commutative:

$$\begin{array}{ccc} \mathbb{R}\times M & \xrightarrow{\Phi} & \mathbb{R}\times M \\ \downarrow p & & \downarrow p \\ \mathbb{R} & \xrightarrow{\phi} & \mathbb{R} \end{array}$$

Let $\mathrm{Aut}(p)$ be the group of automorphisms of $p$. The diffeomorphism $\phi$ is completely determined by $\Phi$; hence, $\mathrm{Aut}(p)$ is a subgroup in $\mathrm{Diff}(\mathbb{R}\times M)$. We denote by $J^r(\mathbb{R},M)$ the bundle of jets of order $r$ of (local) sections of the submersion $p$ and by $\Phi^{(r)}\colon J^r(\mathbb{R},M)\to J^r(\mathbb{R},M)$ the $r$-th jet prolongation of $\Phi$, i.e.,

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http://dx.doi.org/10.1016/j.jmaa.2017.08.057
0022-247X/

$$\Phi^{(r)}(j^r_x\sigma) = j^r_{\phi(x)}\left(\Phi\circ\sigma\circ\phi^{-1}\right), \quad \forall j^r_x\sigma\in J^r(\mathbb{R},M).$$

Let $\mathcal{L},\bar{\mathcal{L}}\colon J^1(\mathbb{R},M)\simeq\mathbb{R}\times TM\to\mathbb{R}$ be two first-order (time dependent) Lagrangians on $M$. The Lagrangians densities $\mathcal{L}dx$, $\bar{\mathcal{L}}dx$ are said to be equivalent if there exists $\Phi\in\mathrm{Aut}(p)$ such that, $(\Phi^{(1)})^*(\mathcal{L}dx)=\bar{\mathcal{L}}dx$, i.e., $\bar{\mathcal{L}}dx=(\mathcal{L}\circ\Phi^{(1)})\phi^*(dx)$, or even,

$$\bar{\mathcal{L}} = \frac{d\phi}{dx}\left(\mathcal{L}\circ\Phi^{(1)}\right). \tag{1}$$

The equivalence problem is one of the basic questions in the Calculus of Variations and has been dealt with in several works. The notion of equivalence itself has different interpretations. Equivalence up to an automorphism of $p$ (also known as *fiber preserving* equivalence) is probably the most natural one and is the approach we follow in this article, although there are other possibilities, such as equivalence under the group $\mathrm{Diff}(\mathbb{R}\times M)$ (point transformations) or even larger groups. The usual tool used to solve the equivalence problem in the literature has been the Cartan method. This procedure is quite algorithmic but the computations become early too complex. However, partial results have been obtained: Equivalence of quadratic Lagrangians in [8] or [1] (with respect to the Euler-Lagrange equations in the last case), equivalence of Lagrangian for $M=\mathbb{R}$ in [5] and [9], or the equivalence problems for field theory Lagrangians defined in the plane $\mathbb{R}^2$ and $M=\mathbb{R}$ in [4]. Our results are obtained for an arbitrary manifold $M$, agreeing with the previous results in the particular case $M=\mathbb{R}$. For that, we follow a different method connected with the notion of invariant and infinitesimal transformations. In this paper, we obtain the complete determination of the set of invariants up to order two. The second order invariant is also exploited from a geometrical point of view to give some higher order invariants in connection with the notion of metric invariant. The intrinsic-geometric interpretation of all these objects sheds light to the problem of determination of a complete set of invariants of arbitrary degree.

We now introduce the notion of invariant. We first note that sections of the projection

$$\begin{aligned} &q\colon J^1(\mathbb{R},M)\times\mathbb{R}\to J^1(\mathbb{R},M),\\ &q(j^1_x\sigma,\lambda)=j^1_x\sigma, \end{aligned}$$

correspond bijectively with functions in $C^\infty(J^1(\mathbb{R},M))$, i.e., with first-order Lagrangians on the fibred manifold $p\colon\mathbb{R}\times M\to\mathbb{R}$. The section $s_{\mathcal{L}}\colon J^1(\mathbb{R},M)\to J^1(\mathbb{R},M)\times\mathbb{R}$ of $q$ corresponding to $\mathcal{L}\in C^\infty(J^1(\mathbb{R},M))$ is given by $s_{\mathcal{L}}(j^1_x\sigma)=(j^1_x\sigma,\mathcal{L}(j^1_x\sigma))$. In what follows $s_{\mathcal{L}}$ and $\mathcal{L}$ will be identified.

Let $\tilde{\Phi}^{(1)}\colon J^1(\mathbb{R},M)\times\mathbb{R}\to J^1(\mathbb{R},M)\times\mathbb{R}$ be the automorphism of $q$ defined as follows:

$$\tilde{\Phi}^{(1)}(j^1_x\sigma,\lambda)=(\Phi^{(1)}(j^1_x\sigma),(\phi')^{-1}\lambda), \tag{2}$$

where $\phi'=d\phi/dx$. If $(\Psi,\psi)$ is another automorphism of $p$, then

$$((\Phi\circ\Psi)^*dx)_x=\Psi^*(\Phi^*dx)_x=\psi^*(\phi'dx)_x=\phi'\psi'(dx)_x.$$

Hence

$$\begin{aligned}\tilde{\Phi}^{(1)}\left(\tilde{\Psi}^{(1)}(j^1_x\sigma,\lambda)\right) &= \tilde{\Phi}^{(1)}(\Psi^{(1)}(j^1_x\sigma),(\psi')^{-1}\lambda)\\ &=\left(\Phi^{(1)}\left(\Psi^{(1)}(j^1_x\sigma)\right),(\phi')^{-1}(\psi')^{-1}\lambda\right)\\ &=\left((\Phi\circ\Psi)^{(1)}(j^1_x\sigma),(\phi'\psi')^{-1}\lambda\right).\end{aligned}$$

In other words, $\widetilde{(\Phi\circ\Psi)}^{(1)}=\tilde{\Phi}^{(1)}\circ\tilde{\Psi}^{(1)}$.