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Improved Hardy and Rellich inequalities on nonreversible Finsler manifolds

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ABSTRACT

In this paper, we study the sharp constants of quantitative Hardy and Rellich inequalities on nonreversible Finsler manifolds equipped with arbitrary measures. In particular, these inequalities can be globally refined by adding remainder terms like the Brezis–Vázquez improvement, if Finsler manifolds are of strictly negative flag curvature, vanishing S-curvature and finite uniformity constant. Furthermore, these results remain valid when Finsler metrics are reversible.

1. Introduction

Let Ω be either $\mathbb{R}^n$ or a bounded domain in $\mathbb{R}^n$ containing 0. Then the Hardy and Rellich inequalities can be stated as follows, respectively: for all $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u|^2 dx \geq \frac{(n-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx, \text{ if } n \geq 3, \tag{1.1}$$

$$\int_\Omega |\Delta u|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx, \text{ if } n \geq 5. \tag{1.2}$$

Here both the constants $\frac{(n-2)^2}{4}$ and $\frac{n^2(n-4)^2}{16}$ are sharp but never archived, which inspires one to improve these inequalities by adding some nonnegative correction terms to the right-hand side of (1.1) and (1.2). In fact, Brezis–Vázquez in [7] showed that if Ω is bounded, then there exists a constant $C_\Omega > 0$ such that

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$$\int_\Omega |\nabla u|^2 dx \geq \frac{(n-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx + C_\Omega \int_\Omega u^2 dx,$$

while Gazzola–Grunau–Mitidieri [14] proved that there are positive constants C_1 and C_2 so that

$$\int_\Omega |\Delta u|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx + C_1 \int_\Omega \frac{u^2}{|x|^2} dx + C_2 \int_\Omega u^2 dx.$$

We refer to [1,2,6,7,14,22] and references therein for more improvements of (1.1) and (1.2).

Hardy and Rellich inequalities have also been investigated in the Riemannian setting. Carron [9] obtained some weighted L^2 Hardy inequalities on complete, non-compact Riemannian manifolds. Afterwards, Kombe–Özaydin [15] and Yand–Su–Kong [23] established some weighted L^p Hardy and Rellich inequalities and presented various Brezis–Vázquez type improvements on complete Riemannian manifolds.

Finsler geometry is Riemannian geometry without quadratic restriction. In order to present the corresponding results in Finsler geometry, we introduce some notations and notions first.

Let (M, F) be a Finsler manifold. The Finsler metric F is called reversible if $F(x, -y) = F(x, y)$, otherwise it is called nonreversible (or irreversible). Clearly, a Riemannian metric is always a reversible Finsler metric. Let $d_F : M \times M \to \mathbb{R}$ denote the distance function induced by F. It is remarkable that d_F is usually asymmetric, that is, $d_F(p, q) \neq d_F(q, p)$. Thus, given a point $p \in M$, we define forward and backward distance functions (with respect to p) as follows:

$$\rho_+(x) := d_F(p, x),\ \ \rho_-(x) := d_F(x, p),\ \ x \in M.$$

For a reversible metric, ρ_+ coincides with ρ_- and hence, we use $\rho(x)$ to denote it.

As far as we know, Kristály–Repovš [17] first considered weighted Hardy and Rellich inequalities in the Finsler setting. More precisely, let (M, F) be an n-dimensional reversible Finsler–Hadamard manifold, equipped with the Busemann–Hausdorff measure $d\mathfrak{m}_{BH}$, with nonpositive flag curvature $\mathbf{K} \leq k \leq 0$ and vanishing S-curvature of $d\mathfrak{m}_{BH}$. Then the Hardy inequality in [17] can be stated as follows: for any $\beta \in \mathbb{R}$ with $n - 2 > \beta$ and all $u \in C_0^\infty(M)$,

$$\begin{aligned}\int_M \frac{F^2(\nabla u(x))}{\rho^\beta(x)}\, d\mathfrak{m}_{BH}(x) &\geq \frac{(n-2-\beta)^2}{4} \int_M \frac{u^2(x)}{\rho^{\beta+2}(x)}\, d\mathfrak{m}_{BH}(x) \\ &\quad + \frac{(n-1)(n-\beta-2)}{2} \int_M \frac{u^2(x)}{\rho^{\beta+2}(x)} D_k(\rho(x))\, d\mathfrak{m}_{BH}(x),\end{aligned} \tag{1.3}$$

where

$$D_k(t) := \begin{cases} 0, & \text{if } k = 0, \\ \sqrt{|k|}\, t \coth(\sqrt{|k|}\, t) - 1, & \text{if } k < 0. \end{cases}$$

Furthermore, the Rellich inequality on reversible Finsler–Hadamard manifolds in [17] reads, for any $\beta \in \mathbb{R}$ with $-2 < \beta < n - 4$ and every $u \in C_0^\infty(M)$ with $G_F^\beta(u) = 0$,

$$\begin{aligned}\int_M \frac{(\Delta u(x))^2}{\rho^\beta(x)}\, d\mathfrak{m}_{BH}(x) &\geq \frac{(n+\beta)^2(n-4-\beta)^2}{16} \int_M \frac{u^2(x)}{\rho^{\beta+4}(x)}\, d\mathfrak{m}_{BH}(x) \\ &\quad + \frac{(n-1)(n-2)(n+\beta)(n-\beta-4)}{4} \int_M \frac{u^2(x)}{\rho^{\beta+4}(x)} D_k(\rho(x))\, d\mathfrak{m}_{BH}(x),\end{aligned} \tag{1.4}$$