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A note on Hurwitz's inequality ☆

Julià Cufí, Eduardo Gallego *, Agustí Reventós

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

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A B S T R A C T

Given a simple closed plane curve Γ of length L enclosing a compact convex set K of area F, Hurwitz found an upper bound for the isoperimetric deficit, namely $L^2 - 4\pi F \leq \pi|F_e|$, where F_e is the algebraic area enclosed by the evolute of Γ. In this note we improve this inequality finding strictly positive lower bounds for the deficit $\pi|F_e| - \Delta$, where $\Delta = L^2 - 4\pi F$. These bounds involve either the visual angle of Γ or the pedal curve associated to K with respect to the Steiner point of K or the $\mathcal{L}^2$ distance between K and the Steiner disk of K. For compact convex sets of constant width Hurwitz's inequality can be improved to $L^2 - 4\pi F \leq \frac{4}{9}\pi|F_e|$. In this case we also get strictly positive lower bounds for the deficit $\frac{4}{9}\pi|F_e| - \Delta$. For each established inequality we study when equality holds. This occurs for those compact convex sets being bounded by a curve parallel to an hypocycloid of 3, 4 or 5 cusps or the Minkowski sum of this kind of sets.

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1. Introduction

Let Γ be a simple closed plane curve of length L enclosing a region of area F. The classical isoperimetric inequality states that

$$L^2 - 4\pi F \geq 0,$$

with equality attained only for a circle.

In the case that Γ bounds a convex set K, Hurwitz ([5]) established a kind of reverse isoperimetric inequality, namely

$$L^2 - 4\pi F \leq \pi|F_e|, \tag{1}$$

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* Corresponding author.
E-mail addresses: jcufi@mat.uab.cat (J. Cufí), egallego@mat.uab.cat (E. Gallego), agusti@mat.uab.cat (A. Reventós).

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where F_e is the algebraic area ($F_e \leq 0$) enclosed by the evolute of Γ. We recall that the evolute of a curve is the envelope of its normal lines or equivalently the locus of the centers of curvature of the curve. Moreover equality holds in (1) if and only if Γ is a circle or a curve parallel to an astroid.

The goal of this note is to improve Hurwitz's inequality (1) finding strictly positive lower bounds for the *Hurwitz deficit* $\pi|F_e| - \Delta$, where $\Delta = L^2 - 4\pi F$ is the *isoperimetric deficit*. These bounds involve either the visual angle of Γ or the pedal curve associated to K with respect to the Steiner point of K or the $\mathcal{L}^2$ distance between the support function of K and the support function of the Steiner disk of K.

Hurwitz's inequality (1) can be improved without introducing new quantities for some special compact sets. For instance, if K has constant width one gets

$$L^2 - 4\pi F \leq \frac{4}{9}\pi|F_e|,$$

as shown in Theorem 5.1.

For the general case we prove in Theorem 4.1 the inequality

$$\pi|F_e| - \Delta \geq \frac{5}{4}L^2 + 5 \int\limits_{P \notin K} (\omega - \sin\omega - \frac{2}{3}\sin^3\omega)dP,$$

where ω is the visual angle of Γ from P, that is the angle between the tangents from P to Γ, and dP the area measure. For the case of constant width Theorem 5.3 asserts that

$$\frac{4}{9}\pi|F_e| - \Delta \geq \frac{64}{9} \int\limits_{P \notin K} \left(\omega - 2\sin\omega + \sin 2\omega - \frac{1}{4}\sin 4\omega - \sin^3\omega\right) dP.$$

In both cases the quantities in the right hand side are strictly positive except when the left hand side vanishes.

In terms of the area A of the pedal curve associated to the compact strictly convex set K, with respect to its Steiner point, we prove in Theorem 4.3

$$\pi|F_e| - \Delta \geq \frac{40}{9}\left(\pi(A - F) + \frac{2}{3}L^2 - \frac{8}{9}\int\limits_{P \notin K} \sin^3\omega \, dP\right).$$

When K has constant width we obtain (Corollary 5.4)

$$\pi|F_e| - \Delta \geq \frac{40}{9}\pi(A - F).$$

In both cases the lower bounds for the positive Hurwitz deficit are strictly positive.

For each established inequality we study when equality holds. This occurs for those compact convex sets being bounded by a curve parallel to an hypocycloid of 3, 4 or 5 cusps or the Minkowski sum of this kind of sets.

2. Preliminaries

2.1. Convex sets and support function

A set $K \subset \mathbb{R}^2$ is *convex* if it contains the complete segment joining every two points in the set. We shall consider nonempty compact convex sets. The *support function* of K is defined as

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