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Navier–Stokes equations with external forces in Lorentz spaces and its application to the self-similar solutions

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ABSTRACT

We show existence theorem of global mild solutions with small initial data and external forces in Lorentz spaces with scaling invariant norms. If the initial data have more regularity in another scaling invariant class, then our mild solution is actually the strong solution. The result on local existence of solutions for large data is also discussed. Our method is based on the maximal regularity theorem on the Stokes equations in Lorentz spaces. Then we apply our theorem to prove existence of self-similar solutions provided both initial data and external forces are homogeneous functions. Since we construct the global solution by means of the implicit function theorem, as a byproduct, its stability with respect to the given data is necessarily obtained.

0. Introduction

Let us consider the Cauchy problem of the Navier–Stokes equations in $\mathbb{R}^n$, $n \geq 2$;

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = f \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u|_{t=0} = a \quad \text{in } \mathbb{R}^n, \end{cases} \tag{N–S}$$

where $u = u(x,t) = (u_1(x,t), \cdots, u_n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the unknown pressure at the point $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and the time $t \in (0,\infty)$, while $a = a(x) = (a_1(x), \cdots, a_n(x))$ and $f = f(x,t) = (f_1(x,t), \cdots, f_n(x,t))$ are the given initial data of velocity and the given external force, respectively. In this paper, we prove the existence of global mild and strong solutions to (N–S) for small initial data $a \in L^{n,\infty}(\mathbb{R}^n)$ and small external force $f \in L^{s,\infty}(0,\infty; L^{q,\infty}(\mathbb{R}^n))$ with

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$2/s+n/3=3$, $n/3<q<\infty$, where $L^{r,\infty}$ denotes the Lorentz space. It is well-known that (N–S) is invariant under such a change of scaling as $u_\lambda(x,t)=\lambda u(\lambda x,\lambda^2 t)$ and $p_\lambda(x,t)=\lambda^2 p(\lambda x,\lambda^2 t)$ for all $\lambda>0$. The Banach space $\mathcal{Y}$ of functions with the space and time variables with the norm $\|\cdot\|_{\mathcal{Y}}$ is called *scaling invariant* to (N–S) if it holds that $\|u_\lambda\|_{\mathcal{Y}}=\|u\|_{\mathcal{Y}}$ for all $\lambda>0$. For instance, in the usual Lebesgue spaces L^q, we see that the scaling invariant space $\mathcal{Y}$ to (N–S) is the Serrin class $L^{s_0}(0,\infty;L^{q_0}(\mathbb{R}^n))$ for $2/s_0+n/q_0=1$ with $n\le q_0\le\infty$. Since the corresponding scaling law to the initial data is like $a_\lambda(x)=\lambda a(\lambda x)$, the suitable Banach space X with the norm $\|\cdot\|_X$ for the initial data should have the property that $\|a_\lambda\|_X=\|a\|_X$ for all $\lambda>0$.

Since the pioneer work of Fujita–Kato [7], many efforts have been made to find such a space X as large as possible. Indeed, Kato [11] and Giga–Miyakawa [8] succeed to find the space $X=L^n(\mathbb{R}^n)$. Later on, Kozono–Yamazaki [13], [14] and Cannone–Planchon [4] extended to $X=L^{n,\infty}(\mathbb{R}^n)$, $\dot{B}^{-1+n/q}_{q,\infty}(\mathbb{R}^n)$ with $n<q<\infty$, where $\dot{B}^s_{q,r}(\mathbb{R}^n)$ denotes the homogeneous Besov space. Introducing the space $\mathcal{PM}^k$ of the pseudo measure defined by $\mathcal{PM}^k=\{a\in\mathcal{S}';\sup_{\xi\in\mathbb{R}^n}|\xi|^k|\hat{a}(\xi)|<\infty\}$, Cannone–Karch [3] proved that $X=\mathcal{PM}^2$ is a suitable space for $n=3$. The largest space of X was obtained by Koch–Tataru [12] who proved local well-posedness of (N–S) for $a\in X=BMO^{-1}=\dot{F}^{-1}_{\infty,2}(\mathbb{R}^n)$, where $\dot{F}^s_{q,r}(\mathbb{R}^n)$ denotes the homogeneous Triebel–Lizorkin space. Their result [12] seems to be optimal in the sense that continuous dependence of solutions with respect to the initial data breaks down in $X=\dot{B}^{-1}_{\infty,r}(\mathbb{R}^n)$ for $2<r\le\infty$, which was proved by Bourgain–Pavlović [2], Yoneda [23] and Wang [21]. Iwabuchi [10] introduced the modulation space $M^s_{q,r}(\mathbb{R}^n)$ and proved similar ill-posedness to those of [2] and [23] in $X=M^s_{2,r}$ for $s<-1$ and $1\le r<\infty$.

Concerning the external force f, the corresponding scaling law is like $f_\lambda(x,t)=\lambda^3 f(\lambda x,\lambda^2 t)$. However, in comparison with a number of papers on well-posedness with respect to the initial data, there is a little literature for investigating the suitable space Y of external forces satisfying $\|f_\lambda\|_Y=\|f\|_Y$ for all $\lambda>0$. A typical Lebesgue space of Y can be chosen as $Y=L^s(0,\infty;L^q(\mathbb{R}^n))$ for $2/s+n/q=3$. In this direction, Cannone–Planchon [5] treated $f=\mathrm{div}\ F$ with $F\in L^s(0,\infty;L^q(\mathbb{R}^3))$ for $2/s+3/q=2$ with $2/3<q<\infty$. Then Cannone–Karch [3] showed that $Y=C_{\mathrm{w}}(0,\infty;\mathcal{PM}^0)$ is a suitable space for $n=3$. (C_{w} denotes the class of weakly continuous functions.) Their space may be regarded as the variant of $L^\infty(0,\infty;L^{n/3,\infty}(\mathbb{R}^n))$.

In this paper, for the global well-posedness of (N–S), we take such a space Y as the Lorentz space $L^{s,\infty}(0,\infty;L^{q,\infty}(\mathbb{R}^n))$ for $2/s+n/q=3$ with $3/n<q<\infty$ as well as $X=L^{n,\infty}(\mathbb{R}^n)$. Our method is based on the maximal regularity theorem on the Stokes operator $A=-P\Delta$ in $L^{s,\infty}(0,\infty;L^{q,\infty}(\mathbb{R}^n))$ with P denoting by the projection onto the space of solenoidal vector fields, while they [3] made use of analyticity and decay properties of the Stokes semigroup $\{e^{-tA}\}_{t>0}$ in $\mathcal{PM}^k$. Since the maximal regularity theorem enables us to choose $1<s,q<\infty$ arbitrarily, we may take larger spaces than those of [3]. Furthermore, since the homogeneous functions can be handled in the class of Lorentz spaces $L^{q,\infty}(\mathbb{R}^n)$, we show the existence of self-similar solutions to (N–S). It should be also emphasized that our construction of solutions relies on the implicit function theorem which yields necessarily continuous dependence $(a,f)\to u$ from $X\times Y$ to $\mathcal{Y}$. Hence, we prove global well-posedness of (N–S) with small data a and f in our Lorentz spaces. Concerning the existence of local strong solutions to (N–S) for large given data a and f, because of lack of strong continuity of the semi-group $\{e^{-tA}\}_{t>0}$ in $L^{q,\infty}(\mathbb{R}^n)$, we have a certain restriction on a and f. However, our condition may handle arbitrary data $a\in L^n(\mathbb{R}^n)$ and $f\in L^s(0,T;L^{q,\infty}(\mathbb{R}^n))$ for $2/s+n/q=3$ with $3/n<q<\infty$.

1. Results

Before stating our result, let us first recall the Lorentz space $L^{q,\infty}$ on $\mathbb{R}^n$ defined by

$$L^{q,\infty}=\{f:\mathbb{R}^n\to\mathbb{R}^n;\ \|f\|_{L^{q,\infty}}=\sup_{R>0}R\,[\mu\{x\in\mathbb{R}^n:|f(x)|>R\}]^{\frac{1}{q}}<\infty\},\tag{1.1}$$