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Spectral radii of truncated circular unitary matrices

Wenhao Gui [a], Yongcheng Qi [b,*]

[a] Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China
[b] Department of Mathematics and Statistics, University of Minnesota Duluth, MN 55812, USA

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ABSTRACT

Consider a truncated circular unitary matrix which is a p_n by p_n submatrix of an n by n circular unitary matrix by deleting the last $n - p_n$ columns and rows. Jiang and Qi [11] proved that the maximum absolute value of the eigenvalues (known as spectral radius) of the truncated matrix, after properly normalized, converges in distribution to the Gumbel distribution if p_n/n is bounded away from 0 and 1. In this paper we investigate the limiting distribution of the spectral radius under one of the following four conditions: (1). $p_n \to \infty$ and $p_n/n \to 0$ as $n \to \infty$; (2). $(n - p_n)/n \to 0$ and $(n - p_n)/(\log n)^3 \to \infty$ as $n \to \infty$; (3). $n - p_n \to \infty$ and $(n - p_n)/\log n \to 0$ as $n \to \infty$ and (4). $n - p_n = k \geq 1$ is a fixed integer. We prove that the spectral radius converges in distribution to the Gumbel distribution under the first three conditions and to a reversed Weibull distribution under the fourth condition.

1. Introduction

The early study of large random matrices was stimulated by analysis of high-dimensional data. One example is Wishart's [26] investigation on large covariance matrices whose statistical properties are mainly determined by eigenvalues and eigenvectors from the point view of a principal components analysis. Since then, the random matrix theory has been developed very rapidly and found many applications in areas such as heavy-nuclei atoms [25], number theory [16], quantum mechanics [15], condensed matter physics [7], wireless communications [3].

The study of random matrices has greatly been motivated by Tracy and Widom's [22,23] work. They show that the largest eigenvalues of the three Hermitian matrices (Gaussian orthogonal ensemble, Gaussian unitary ensemble and Gaussian symplectic ensemble) converge to some special distributions that are now known as the Tracy–Widom laws. Subsequently, the Tracy–Widom laws have found their applications in the study of problems such as the longest increasing subsequence [1], combinatorics, growth processes,

* Corresponding author.
E-mail address: yqi@d.umn.edu (Y. Qi).

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random tilings and the determinantal point processes (see, e.g., Tracy and Widom [24], Johansson [12] and references therein) and the largest eigenvalues in the high-dimensional statistics (see, e.g., Johnstone [13,14] and Jiang [9]). Some recent research focuses on the universality of the largest eigenvalues of matrices with non-Gaussian entries; see, for example, Tao and Vu [21], Erdős et al. [6] and the references therein.

Consider a non-Hermitian matrix $\mathbf{M}$ with eigenvalues $z_1, \cdots, z_n$. The largest absolute values of the eigenvalues $\max_{1\le j\le n}|z_j|$ is refereed to as the spectral radius of $\mathbf{M}$. The spectral radii of the real, complex and symplectic Ginibre ensembles are investigated by Rider [18,19] and Rider and Sinclair [20], and it is proved that the spectral radius for the complex Ginibre ensemble converges to the Gumbel distribution. This indicates that non-Hermitian matrices exhibit quite different behaviors from Hermitian matrices in terms of the limiting distribution for the largest absolute values of the eigenvalues.

A very recent paper by Jiang and Qi [11] studies the largest radii of three rotation-invariant and non-Hermitian random matrices: the spherical ensemble, the truncation of circular unitary ensemble and the product of independent complex Ginibre ensembles. It is proved in the paper that the spectral radii converge to the Gumbel distribution and some new distributions.

The circular unitary ensemble is an $n \times n$ random matrix with Haar measure on the unitary group, and it is also called Haar-invariant unitary matrix. Let $\mathbf{U}$ be an $n \times n$ circular unitary matrix. The n eigenvalues of the circular unitary matrix $\mathbf{U}$ are distributed over $\{z \in \mathcal{C} : |z| = 1\}$, where $\mathcal{C}$ is the complex plane, and their joint density function is given by

$$\frac{1}{n!(2\pi)^n} \cdot \prod_{1\le j<k\le n} |z_j - z_k|^2;$$

see, e.g., Hiai and Petz [8].

For $n > p \ge 1$, write

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{C}^* \\ \mathbf{B} & \mathbf{D} \end{pmatrix}$$

where $\mathbf{A}$, as a truncation of $\mathbf{U}$, is a $p \times p$ submatrix. Let $z_1, \cdots, z_p$ be the eigenvalues of $\mathbf{A}$. Then their density function is

$$C \cdot \prod_{1\le j<k\le p} |z_j - z_k|^2 \prod_{j=1}^{p} (1 - |z_j|^2)^{n-p-1} \tag{1.1}$$

where C is a normalizing constant. See, e.g., Życzkowski and Sommers [27].

Assume $p = p_n$ depends on n and set $c = \lim_{n\to\infty} \frac{p_n}{n}$. Życzkowski and Sommers [27] show that the empirical distribution of z_i's converges to the distribution with density proportional to $\frac{1}{(1-|z|^2)^2}$ for $|z| \le c$ if $c \in (0, 1)$. Dong et al. [5] prove that the empirical distribution goes to the circular law and the arc law as $c = 0$ and $c = 1$, respectively. See also Diaconis and Evans [4] and Jiang [9,10] and references therein for more results.

Jiang and Qi [11] have proved that the spectral radius $\max_{1\le j\le p}|z_j|$ for the truncated circular unitary ensemble converges to the Gumbel distribution when the dimension of the truncated circular unitary matrix is of the same order as the dimension of the original circular unitary matrix, see Theorem 1 in section 2.

In this paper we consider heavily truncated and lightly truncated circular unitary matrices and investigate the limiting distribution of the spectral radii for those truncated circular unitary matrices. Our results complement that in Jiang and Qi [11].

The rest of the paper is organized as follows. The main results in this paper are given in section 2 and their proofs are provided in section 3.