Doctopic: Complex Analysis ARTICLE IN PRESS YJMAA:21709

J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Fischer decomposition for the symplectic group

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ARTICLE INFO

Article history:
Received 14 June 2017
Available online xxxx
Submitted by R.M. Aron

Keywords:
Fischer decomposition
Quaternionic Clifford analysis

ABSTRACT

We prove the Fischer decomposition for the space of spinor-valued polynomials, defined on Euclidean space of four-fold dimension, in terms of irreducible modules for the symplectic group, consisting of so-called $\mathfrak{osp}(4|2)$-monogenics.

1. Introduction

At the core of this paper are spaces of homogeneous quaternionic monogenic polynomials, i.e. polynomials defined in Euclidean space, the dimension of which is assumed to be a fourfold, taking their values in a Clifford algebra, or subspaces thereof, and which are null solutions of four first order differential operators: a quaternionic Dirac operator and three different conjugates of it. The associated function theory is called *quaternionic Clifford analysis*; it is the most recent branch in the still growing but already well established domain of Clifford analysis.

Standard Clifford analysis is, in its most basic form, a higher dimensional generalisation of holomorphic function theory in the complex plane, and a refinement of harmonic analysis. The fundamental notion in this function theory is that of a monogenic function, i.e. a Clifford algebra valued null solution of the Dirac operator $\underline{\partial} = \sum_{\alpha=1}^{m} e_\alpha \, \partial_{x_\alpha}$, where $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$, which underlies the construction of the real Clifford algebra $\mathbb{R}_{0,m}$. This elliptic version of the Dirac equation, which is the basic field equation for particles with spin $\frac{1}{2}$, is the model par excellence for the first order, elliptic, conformally invariant system of PDEs acting on functions defined in a Euclidean vector space and with values in the basic spinor representation of the corresponding spin group.

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https://doi.org/10.1016/j.jmaa.2017.09.041
0022-247X/

When taking the dimension to be even: $m = 2n$ and considering functions with values in the complex Clifford algebra $\mathbb{C}_{2n}$ or in complex spinor space, *hermitian Clifford analysis* arises as a first refinement of standard Clifford analysis by introducing an additional datum, a so-called complex structure $\mathbb{I}$, i.e. an SO-element squaring to minus the identity, which induces an associated, rotated, Dirac operator $\underline{\partial}_{\mathbb{I}}$. Hermitian monogenic functions then are simultaneous null solutions of the operators $\underline{\partial}$ and $\underline{\partial}_{\mathbb{I}}$; the fundamental group underlying this function theory is the unitary group $\mathrm{U}(n)$.

Quaternionic Clifford analysis is a further refinement of hermitian Clifford analysis, originating from the introduction of a second complex structure $\mathbb{J}$, anti-commuting with the first one $\mathbb{I}$, leading to the Dirac operators $\underline{\partial}$, $\underline{\partial}_{\mathbb{I}}$, $\underline{\partial}_{\mathbb{J}}$ and $\underline{\partial}_{\mathbb{IJ}}$. In a series of papers [5,8,6,7] we have thoroughly studied the fundaments of this function theory, in particular aiming at decomposing spaces of spinor-valued homogeneous polynomials in terms of irreducible representations of the symplectic group $\mathrm{Sp}(p)$. It turns out that in order to obtain $\mathrm{Sp}(p)$-irreducibility in this Fischer decomposition, spaces of so-called $\mathfrak{osp}(4|2)$-monogenic polynomials, a subclass of the quaternionic monogenic polynomials, must be considered, the Lie superalgebra $\mathfrak{osp}(4|2)$ being the Howe dual partner to the symplectic group $\mathrm{Sp}(p)$. This new concept of $\mathfrak{osp}(4|2)$-monogenicity is defined by means of the four, already mentioned, quaternionic Dirac operators and two additional operators: a scalar Euler operator $\mathcal{E}$ underlying the notion of symplectic harmonicity (see [6]) and a multiplication operator P in the Clifford algebra, underlying the decomposition of spinor space $\mathbb{S}$ into symplectic cells $\mathbb{S}^r_s$, which are fundamental irreducible $\mathrm{Sp}(p)$-representations (see [5]).

In [7] we have, a.o., conjectured the Fischer decomposition of the space $\mathcal{P}(\mathbb{R}^{4p};\mathbb{S})$ of spinor-valued polynomials in terms of spaces $\mathcal{S}^r_{a,b}$ of bi-homogeneous $\mathfrak{osp}(4|2)$-monogenic polynomials with values in the symplectic cell $\mathbb{S}^r_r$. However the conjectured form is not completely correct in some particular cases. The aim of the underlying paper is to formulate and prove a corrected version of this Fischer decomposition, which holds in *all* cases, while showing also the $\mathrm{Sp}(p)$-irreducibility of the spaces $\mathcal{S}^r_{a,b}$. The latter is done in the spirit of Howe's invariant theory [11]. To make the paper self-contained we have included a section on hermitian, quaternionic and $\mathfrak{osp}(4|2)$-monogenicity, which is special in the sense that it presents an original point of view on the refinements of Clifford analysis alluded on at the beginning of this introduction, through the concept of symmetry reduction.

2. Hermitian, quaternionic and $\mathfrak{osp}(4|2)$-monogenicity

One way to introduce the refinements embodied in the hermitian and quaternionic monogenic function theories, is by answering the following fundamental question: *what is the interplay between systems of equations and their symmetries*? As mentioned above, classical Clifford analysis is centred around the Dirac equation $\underline{\partial} f(\underline{x}) = 0$ in $\mathbb{R}^m$, and the symmetry group for this equation is the conformal one. There are several approaches possible to explaining the meaning of this symmetry phenomenon. One can for instance use Vahlen matrices, which amounts to treating the conformal symmetry at the group level. Another approach consists in determining the so-called *generalised symmetries* for the Dirac operator and investigating the algebraic structure they generate. For the definition of generalised symmetries we refer to e.g. Miller's seminal work [14] in which the connection between these symmetries and the method of separation of variables was investigated (see also [2]). More recently, higher order (generalised) symmetries of e.g. the Laplace and the Dirac operator also appeared in the framework of higher spin symmetry algebras. For the Laplace operator in $\mathbb{R}^m$ we refer to [9] where also a nice explanation of the connection with these higher spin theories is given, and to e.g. [10,13] for further generalisations.

Definition 1. A linear differential operator φ is a generalised symmetry for the Dirac operator $\underline{\partial}$ if there exists another linear differential operator ψ such that $[\varphi, \underline{\partial}] = \psi\underline{\partial}$. In the case where $\psi(x) = 0$, or $[\varphi, \underline{\partial}] = 0$, one says that φ is a (proper) symmetry.