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On polynomial functions on non-commutative groups

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A B S T R A C T

Let G be a topological group. We investigate relations between two classes of "polynomial like" continuous functions on G defined, respectively, by the conditions 1) $\Delta_h^{n+1} f = 0$ for every $h \in G$, and 2) $\Delta_{h_{n+1}} \Delta_{h_n} \cdots \Delta_{h_1} f = 0$ for every $h_1, \cdots, h_{n+1} \in G$. It is shown that for many (but not all) groups these classes coincide. We consider also Montel type versions of the above conditions – when 1) and 2) hold only for h in a generating subset of G. Our approach is based on the study of the counterparts of the discussed classes for general representations of groups (instead of the regular representation).

1. Introduction

In the works of M. Frechet [7,8], Van der Lijn [24], S. Mazur and W. Orlicz [15] it was shown that the ordinary polynomials on $\mathbb{R}$ or $\mathbb{R}^n$ can be characterized by some conditions (functional equations) which allow one to define and study the analogues of polynomials on commutative groups, semigroups and linear spaces.

In this work we are concentrated on the study of two conditions for scalar functions on an *arbitrary* topological group G (discrete groups are regarded as a special class of topological ones): the Fréchet functional equation

$$\Delta_{h_{n+1}} \Delta_{h_n} \cdots \Delta_{h_1} f = 0, \qquad \text{for all} \quad h_1, \cdots, h_{n+1} \in G \tag{1.1}$$

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and the equation for iterated differences

$$\Delta_h^{n+1} f = 0, \qquad \text{for all} \;\; h \in G. \tag{1.2}$$

Here the difference operator Δ_h is defined as $R_h - 1$ where R_h is the right shift, $R_h f(x) = f(xh)$. It will be shown below (see Remark 3.3) that using left shifts one comes to the same classes of functions.

To fix the notations we say that a function $f : G \to \mathbb{C}$ is a *polynomial of degree at most* n if it satisfies (1.1); furthermore, a function $f : G \to \mathbb{C}$ is said to be a *semipolynomial of degree at most* n if it solves (1.2). We denote these classes of functions by (P_n) and (SP_n), respectively. The elements of $(P) = \bigcup_{n\geq 0}(P_n)$ are called *polynomials*, and the elements of $(SP) = \bigcup_{n\geq 0}(SP_n)$ are called *semipolynomials*. Of course, $(P_n) \subseteq (SP_n)$, for every n. In general the inclusion can be strict (see Example 3.10 below). The fact that these two classes coincide for $G = \mathbb{R}$, was established by Mazur and Orlicz [15]. For an arbitrary commutative group G, the equality $(SP_n) = (P_n)$ – not only in the case of complex-valued functions but for maps to an Abelian group with some restrictions on the latter – was proved by Van der Lijn [24], Djoković [6] (here G can be a semigroup), Szekelyhidi [20] (see also an alternative proof in [23]) and Laczkovich [14] (this paper contains the most general results and systematic treatment of the subject).

We will find a wider variety of groups where the equivalence takes place. For example, it will be shown that $(SP_n) = (P_n)$ if G coincides with G_c, the closure of the subgroup generated by all compact elements, and more generally, if G/G_c is commutative (Corollary 3.14). We will prove that the latter class contains all semisimple and many solvable Lie groups.

The question whether every semipolynomial on an arbitrary group is a polynomial (not necessarily of the same degree), is still open; we answer it (affirmatively) only for semipolynomials of degree 1 (Theorem 3.11).

One more functional class related to polynomials is the class (QP) of *quasipolynomials* – the functions whose shifts generate finite-dimensional subspaces. For $G = \mathbb{R}^d$, quasipolynomials are exactly exponential polynomials, that is the sums of products of polynomials and exponential functions $e^{\langle\lambda,x\rangle}$ with $\lambda \in \mathbb{C}^d$. So in this case (and in many others, for example for all Lie groups) (QP) is much wider than (P). On the other hand there are examples of groups for which the inclusion $(P) \subset (QP)$ fails (see Section 3). We will show that inside the class of quasipolynomials there is no difference between semipolynomials and polynomials.

In Section 2 we study analogues of the discussed classes for general representations of groups and then in Section 3 we deal with functions on groups, applying general results to the regular representation.

In Section 4 we consider the following question: is it sufficient to check the conditions (1) and (2) only for h_i from a topologically generating subset of G?

For $G = \mathbb{R}^n$ the question was affirmatively solved by Montel [16,17] in 1935. Montel's original proofs were tricky, so that in last few years several simpler proofs have been published [1–5]. Furthermore, these results have been connected to the theory of local polynomials [4], and some versions of them have been demonstrated also for other classes of functions, e.g., for exponential polynomials [5].

We will prove that the results on polynomials extend to general groups: if f behaves as a polynomial of degree $\leq m$ when we choose the steps h_i from a generating set of G, then f is a polynomial on G of the same degree. The corresponding statement for commutative G was obtained by M. Laczkovich [14, Lemma 15]. For semipolynomials the situation is more complicated; we get only some partial results and counterexamples.

2. Representations

In general we consider weakly continuous representations on locally convex topological linear spaces. If G is discrete then a topology of the underlying space is not important and we deal with arbitrary representations on linear spaces.