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Original article

Several inequalities For log-convex functions

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Abstract

In this paper, we recall Ostrowski's inequality, Hadamard's inequality and the definition of log-convex functions. We also mention an useful integral identity in the first part of our study. The second part of our study includes new results. We prove new generalizations for log-convex functions. Several new Ostrowski type inequalities have been established and some special cases have been given by choosing $h = 0$ or $x = \frac{a+b}{2}$.

Keywords: log-convex functions; Ostrowski inequality; Hermite–Hadamard inequality; Hölder inequality

1. Introduction

Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, such that $f' \in L[a, b]$ where $a, b \in I$ with $a < b$. If $\left|f'(x)\right| \le M$, then the following inequality holds:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right]. \tag{1.1}$$

This inequality is well known in the literature as the Ostrowski inequality.

The following inequality is well known in the literature as the Hermite–Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

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In [1], Pečarić et al. mentioned log-convex functions as follows:

A function $f : I \to [0, \infty)$ is said to be log-convex or multiplicatively convex if $\log f$ is convex, or, equivalently, for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}.$$

Example 1. The function $f(x) = \frac{1}{x}$, $x \in (0, \infty)$ is log-convex on $(0, \infty)$. The function $f(x) = x^x$, $x > 0$ or $f(x) = e^x + 1$, $x \in \mathbb{R}$, etc.

Many different extensions, generalizations and improvements related to log-convex functions can be found in [1–9]. In order to prove our main results we use the following equality from [10] that is mentioned in [11]:

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b), then this equality holds*

$$\int_a^b f(t)\,dt = (b-a)(1-h)f(x) - (b-a)(1-h)\left(x - \frac{a+b}{2}\right) f'(x) + h\frac{b-a}{2}(f(a) + f(b)) - \frac{h^2(b-a)^2}{8}\left(f'(b) - f'(a)\right) + \int_a^b K(x,t) f''(t)dt$$

for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$ and $h \in [0, 1]$. Here $K : [a, b]^2 \to \mathbb{R}$

$$K(x,t) = \begin{cases} \dfrac{1}{2}\left[t - \left(a + h\dfrac{b-a}{2}\right)\right]^2, & \text{if } t \in [a, x] \\ \dfrac{1}{2}\left[t - \left(b - h\dfrac{b-a}{2}\right)\right]^2, & \text{if } t \in (x, b]. \end{cases}$$

The main purpose of this paper is to give some new integral inequalities of Ostrowski type for logarithmically convex functions by using the above lemma.

2. Main results

Let us start our first result:

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b). If $\left|f''\right|$ is* log*-convex, the following inequality holds for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$:*

$$\left| \int_a^b f(t)\,dt - (b-a)(1-h)f(x) + (b-a)(1-h)\left(x - \frac{a+b}{2}\right) f'(x) - h\frac{b-a}{2}(f(a) + f(b)) + \frac{h^2(b-a)^2}{8}\left(f'(b) - f'(a)\right) \right|$$

$$\leq \frac{1}{2}\left[\left(\frac{\left|f''(a)\right|^x}{|f''(x)|^a} \right)^{\frac{1}{x-a}} \tau_1 + \left(\frac{\left|f''(x)\right|^b}{|f''(b)|^x} \right)^{\frac{1}{b-x}} \tau_2 \right]$$

where

$$\tau_1 = \left[x - \left(a + h\frac{b-a}{2}\right)\right]^2 \frac{T^x}{\ln T} - h^2\frac{(b-a)^2}{4}\frac{T^a}{\ln T} - \frac{2}{(\ln T)^2}\left[x - \left(a + h\frac{b-a}{2}\right)\right] T^x + h\frac{b-a}{(\ln T)^2} T^a + \frac{2}{(\ln T)^3}\left[T^x - T^a\right],$$

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