ARTICLE IN PRESS

Available online at www.sciencedirect.com
ScienceDirect

Transactions of A. Razmadze Mathematical Institute ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

Weighted Hardy-type inequalities involving convex function for fractional calculus operators

Sajid Iqbal[a,*], Josip Pečarić[b], Lars-Erik Persson[c,d], Zivorad Tomovski[e]

[a] Department of Mathematics, University of Sargodha (Sub-Campus Mianwali), Mianwali, Pakistan
[b] Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
[c] Department of Mathematics, Luleå University of Technology, SE-971 87, Luleå, Sweden
[d] UIT The Artic University of Norway, P.O. Box 385, N-8505, Narvik, Norway,
[e] Faculty of Mathematics and Natural Sciences, Gazi Baba bb, 1000 Skopje, Macedonia

Received 29 August 2017; received in revised form 5 December 2017; accepted 14 December 2017
Available online xxxxx

Abstract

The aim of this paper is to establish some new weighted Hardy-type inequalities involving convex and monotone convex functions using Hilfer fractional derivative and fractional integral operator with generalized Mittag-Leffler function in its kernel. We also discuss one dimensional cases of our related results. As a special case of our general results we obtain the results of Iqbal et al. (2017). Moreover, the refinement of Hardy-type inequalities for Hilfer fractional derivative is also included.

Keywords: Convex function; Kernel; Hilfer fractional derivatives; Fractional integral

1. Introduction

Fractional calculus deals with the study of fractional order integral and derivative operators calculus and have been of great importance during the last few decades. Oldham and Spanier [1] published their fundamental work in their book in 1974 and Podlubny [2] publication from 1999, which deals principally with fractional differential equations. For further details and literature about the fractional calculus we refer to [3–5] and the references cited therein. Numerous mathematicians obtained new Hardy-type inequalities for different fractional integrals and fractional derivatives. For details we refer to [6–12].

The general theory for the Hardy-type inequalities has attracted a lot of attention during a long time, see e.g. the books [13–15] and the reference therein. One reason is that such results are of special interest for technical sciences.

* Corresponding author.
E-mail addresses: dr.sajid@uos.edu.pk (S. Iqbal), pecaric@element.hr (J. Pečarić), larserik@ltu.se (L.-E. Persson), tomovski@pmf.ukim.mk (Z. Tomovski).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

https://doi.org/10.1016/j.trmi.2017.12.001

Especially actions of kernels operators of type (1.2) and (1.3) are important since the kernel $k(x, y)$ represent unit impulse answers in systems which need not to be time invariant ($f(y)$ and $g(x)$ represent the “insignals” and “outsignals” respectively). Some current knowledge can be found in Section 7.5 of the new 2017 book [15] by Kufner, Persson and Samko, see also the related review article [16]. But still there are many open questions in this area, see e.g. those pointed out in [15, Section 7.5]. In this paper we present some new results concerning Hardy-type inequalities not covered by the literature mentioned above.

The following definitions are presented in [17].

Definition 1.1. Let I be an interval in $\mathbb{R}$. A function $\Phi : I \to \mathbb{R}$ is called convex if

$$\Phi(\lambda x + (1-\lambda)y) \leq \lambda\Phi(x) + (1-\lambda)\Phi(y), \tag{1.1}$$

for all points $x, y \in I$ and all $\lambda \in [0, 1]$. The function Φ is strictly convex if inequality (1.1) holds strictly for all distinct points in I and $\lambda \in (0, 1)$.

Definition 1.2. Let $\Phi : I \longrightarrow \mathbb{R}$ be a convex function, then the sub-differential of Φ at x, denoted by $\partial\Phi(x)$, is defined as

$$\partial\Phi(x) = \{\alpha \in \mathbb{R} : \Phi(y) - \Phi(x) - \alpha(y - x) \geq 0, \ y \in I\}.$$

Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with positive σ-finite measures. Let $U(f)$ denote the class of functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x, y) f(y) d\mu_2(y), \tag{1.2}$$

and A_k be an integral operator defined by

$$(A_k f)(x) := \frac{g(x)}{K(x)} = \frac{1}{K(x)} \int_{\Omega_2} k(x, y) f(y) d\mu_2(y), \tag{1.3}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, $f : \Omega_2 \to \mathbb{R}$ is measurable function and

$$0 < K(x) := \int_{\Omega_2} k(x, y) d\mu_2(y), \qquad x \in \Omega_1. \tag{1.4}$$

The following theorem was given in [18] and [19] (see also [20]).

Theorem 1.3. *Let $0 < p \leq q < \infty$, or $-\infty < q \leq p < 0$, $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, K be defined on Ω_1 by (1.4) and that the function $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on Ω_1 for each $y \in \Omega_2$, and that v is defined on Ω_2 by*

$$v(y) := \left(\int_{\Omega_1} u(x)\left(\frac{k(x, y)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int I$, then the inequality

$$\begin{aligned} &\left(\int_{\Omega_2} v(y)\Phi\left(f(y)\right) d\mu_2(y)\right)^{\frac{q}{p}} - \int_{\Omega_1} u(x)[\Phi\left((A_k f)(x)\right)]^{\frac{q}{p}} d\mu_1(x) \\ &\geq \frac{q}{p} \int_{\Omega_1} \frac{u(x)}{K(x)} \Phi^{\frac{q}{p}-1}((A_k f)(x)) \int_{\Omega_2} k(x, y) r(x, y) d\mu_2(y) d\mu_1(x) \end{aligned} \tag{1.5}$$

holds for all measurable functions $f : \Omega_2 \to I$, where A_k is defined by (1.3) and $r : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a non-negative function defined by

$$r(x, y) = ||\Phi(f(y)) - \Phi((A_k f)(x))| - |\varphi((A_k f)(x))||f(y) - (A_k f)(x)||. \tag{1.6}$$