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Original article

Aitken type methods with high efficiency

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Abstract

In this paper, we study the iterative method of Aitken type for solving the non-linear equations, in which the interpolation nodes are controlled by variant of Newton method or by a general method of order p. By combining such methods with a generalized secant method, it is shown that the order of convergence can be increased to as high as desired and also in the limiting case efficiency of the method is 2. Several numerical examples are provided in support of the theoretical results.

Keywords: Non-linear equations; Newton method; Aitken type method; Generalized secant method

1. Introduction

Non-linear equations are encountered in all branch of science and engineering. It is hardly possible to solve such equations analytically and therefore iterative methods are employed. For a given non-linear equation

$$f(x) = 0,$$

a very well known method widely used is the Newton method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

which is quadratically convergent. There have been several ways by which the order of convergence can be increased. Recently, in [1], Păvăloiu and Cătinaş obtained and studied the following Aitken method:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(y_n)}{f'(y_n)},$$

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$$x_{n+1} = z_n - \frac{f(z_n)}{[y_n, z_n; f]}. \tag{1.1}$$

Along with other considerations, it was proved in [1] that the method (1.1) is of order 6 with efficiency index 1.431 which is higher than the Newton method or the standard Aitken method.

Recently, McDougall and Wotherspoon in [2] gave the following modification of Newton's method:

$$\begin{aligned} x_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}[x_{n-1} + x_{n-1}^*]\right)} \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}[x_n + x_n^*]\right)}. \end{aligned} \tag{1.2}$$

They proved that the above method (1.2) has order of convergence $1+\sqrt{2} \approx 2.4$ and requires two functions evaluation per iteration so that the efficiency becomes 1.5537. In this paper, to begin with, we propose the following method in which the Newton iterates in (1.1) are replaced by (1.2):

$$\begin{aligned} y_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}[x_{n-1} + y_{n-1}^*]\right)} \\ y_n &= x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}[x_n + y_n^*]\right)} \\ z_n^* &= y_n - \frac{f(y_n)}{f'\left(\frac{1}{2}[y_{n-1} + z_{n-1}^*]\right)} \\ z_n &= y_n - \frac{f(y_n)}{f'\left(\frac{1}{2}[y_n + z_n^*]\right)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{[y_n, z_n; f]}. \end{aligned} \tag{1.3}$$

We prove that the order of convergence of the method (1.3) is 6.76137 and efficiency is 1.4655 higher than the method (1.1). This is done in Section 2.

Also in Section 2, we study a method more general than (1.1) or (1.2). We replace, in (1.1), the Newton iterates by the iterates of any arbitrary method. Let $\phi(x)$ be an iterative function such that the method

$$x_{n+1} = \phi(x_n)$$

is of order p. We propose the following generalized Aitken-type method:

$$\begin{aligned} y_n &= \phi(x_n), \\ z_n &= \phi(y_n), \\ x_{n+1} &= z_n - \frac{f(z_n)}{[y_n, z_n; f]}. \end{aligned} \tag{1.4}$$

We prove, in Section 2, that the method (1.4) is of order $p^2 + p$. This strategy would enable to produce an iterative method of any desired order. We demonstrate it with the help of certain examples.

There have been several methods which are based on approximations of integrals. In this direction Weerakoon and Fernando [3] obtained the following third order method:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}. \end{aligned} \tag{1.5}$$

Several authors have obtained similar methods, see, e.g., [3–5]. It is noted that the method (1.5) will not proceed if at any iterate $f'(x_n) = 0$. To overcome this problem, on the lines of Wu [6], in [7], the following modification of (1.5) was proposed and studied:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n) - \lambda f(x_n)}$$