Available online at www.sciencedirect.com
ScienceDirect

Transactions of A. Razmadze Mathematical Institute 172 (2018) 238–250

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

Study residuated lattice via some elements

R. Tayebi Khorami[a], A. Borumand Saeid[b,*]

[a] *Department of Mathematics, Ahvaz Branch, Islamic Azad university, Ahvaz, Iran*
[b] *Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran*

Received 14 June 2017; received in revised form 3 December 2017; accepted 4 February 2018
Available online 31 March 2018

Abstract

In this paper, the notions of distributive, standard and neutral elements in residuated lattices were introduced and relationships between them were investigated. Also we study the sets of distributive, standard and neutral elements in residuated lattices. Then we show that under some conditions, the sets of distributive, standard and neutral elements in residuated lattices become a MTL-algebra. Finally, special elements of type 2 in residuated lattices were introduced.

MSC: 03B47; 03G25

Keywords: Residuated lattice; MTL-algebra; BL-algebra; Distributive element; Standard element; Neutral element

1. Introduction and preliminaries

The concept of residuated lattices was introduced by M. Ward and R. P. Dilworth [1] as a generalization of the structure of t he set of ideals of a ring. These algebras are a common structure among algebras associated with logical systems. The residuated lattices have interesting algebraic and logical properties [2–5]. The main example of residuated lattices related to logic is BL-algebras. A basic logic algebra (BL-algebra for short) is an important class of logical algebras introduced by H'ajek [6] in order to provide an algebraic proof of the completeness of "Basic Logic" (BL for short). MV-algebras introduced by Chang [7] in 1958 are the most known classes of BL-algebras.

The concepts of distributive, standard and neutral elements introduced in lattices by O. Ore [8], G. Gratzer [9] and G. Birkhoff [10], respectively and have been extended to trellises by S. B. Rai in [11].

We decide to generalize this concepts to residuated lattices. In this paper, we introduce the notions of distributive, standard and neutral elements in residuated lattices and verify relationships between them. Also we study the sets of $Dis(L)$, $St(L)$ and $Neu(L)$. Then, we study relationships between distributive, standard and neutral elements with

* Corresponding author.
E-mail addresses: r.t.khorami@gmail.com (R.T. Khorami), arsham@mail.uk.ac.ir (A.B. Saeid).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

https://doi.org/10.1016/j.trmi.2018.02.003

some other special elements, likeness dense, boolean, node and regular elements in residuated lattices. Finally, we study image of distributive, standard and neutral elements under a homomorphism.

In this section, we recall some definitions and results about residuated lattices which are used in the sequel.

Definition 1.1 ([6]). A residuated lattice is an algebra $(L, \vee, \wedge, \odot, \longrightarrow, 0, 1)$ of type (2, 2, 2, 2, 0, 0) such that:
(RL1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
(RL2) $(L, \odot, 1)$ is an abelian monoid,
(RL3) $x \odot z \leq y$ if and only if $z \leq x \to y$, for all $x, y, z \in L$.

Definition 1.2 ([6,12,13]). Let L be a residuated lattice and $a \in L$,
(a) a is called idempotent iff $a^2 = a$,
(b) a is called nilpotent iff there exists a natural number n, such that $a^n = 0$,
(c) a is called dense iff $a^* = 0$, where $x^* = x \to 0$,
(d) a is called regular iff $a^{**} = a$ and $x^* \odot (x^* \to x) = 0$,
(e) a is called boolean iff there exists $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$,
(f) a is called node iff for every filter F of L, $[a) \subseteq F$ or $F \subseteq [a)$.

Theorem 1.3 ([6]). *In any residuated lattice* $(L, \vee, \wedge, \odot, \to, 0, 1)$ *the following properties are valid:*
(1) $1 \to x = x$,
(2) $x \to x = 1$,
(3) $x \odot y \leq x, y$, *so* $x \odot y \leq x \wedge y$,
(4) $y \leq x \to y$,
(5) $x \leq y \Leftrightarrow x \to y = 1$,
(6) if $x \to y = y \to x = 1$, *then* $x = y$,
(7) $x \to 1 = 1$,
(8) $0 \to x = 1$,
(9) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
for all $x, y, z, \in L$.

Definition 1.4 ([6]). A residuated lattice L is called MTL-algebra if the following property is valid:
(BL5) $(x \to y) \vee (y \to x) = 1$, for all $x, y \in L$.

Definition 1.5 ([6]). A MTL-algebra L is called BL-algebra if the following property is valid:
(BL4) $x \odot (x \to y) = x \wedge y$, for all $x, y \in L$.

Definition 1.6 ([6]). A nonempty subset F of residuated lattice L is called a filter of L if F satisfies the following conditions:
$(F1)$ if $x \in F$, $x \leq y$ and $y \in L$, then $y \in F$,
$(F2)$ $x \odot y \in F$ for every $x, y \in F$ that is, F is a subsemigroup of L.

Definition 1.7 ([14]). An element a of a lattice L is called
(1) distributive, if $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$,
(2) standard, if $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$,
(3) neutral, if $(a \wedge x) \vee (a \wedge y) \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) \wedge (x \vee y)$,
for all $x, y \in L$.

The concepts of dually distributive and dually standard elements are obtained by dualizing (1) and (2) respectively. The notion of a neutral element is self-dual.

2. On distributive elements in residuated lattices

From now on $L = (L, \vee, \wedge, \odot, \to, 0, 1)$ is a residuated lattice unless otherwise specified.