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Original article

Stationary distribution and extinction of a three-species food chain stochastic model

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Abstract

In this paper, we investigate long-time behaviour of a stochastic three-species food chain model. By Markov semigroups theory, we prove that the densities of this model can converge to an invariant density or can converge weakly to a singular measure in L^1 under appropriate conditions. Further, several sufficient conditions for the extinction of the three species were obtained. Finally, numerical simulations are carried out to illustrate our theoretical results.

Keywords: Food chain; Stationary distribution; Markov semigroups; Extinction

1. Introduction

The dynamical relationship of three species predator–prey systems has long been one of the hot topics in mathematical biology. To clarify the short-term or long-term behaviour of ecosystems, it is essential to understand the interacting dynamics of three species food chain models. Since 1970s, there have been some interesting results on investigating the dynamics of three species predator–prey systems [1–6]. In particular, Krikorian [4] considered the Volterra predator–prey model in the three species case and to say as much as possible about global properties of its solution. Zhou [3] investigated the existence and global stability of the positive periodic solutions of delayed discrete food chains with omnivory. Hsu [5] considered a three species Lotka–Volterra food web model with omnivory which is defined as feeding on more than one trophic level. A famous three-species food chain model [4] can be expressed

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as follows:

$$\begin{cases} \dot{x}_1(t) = x_1(t)[a_1 - b_{11}x_1(t) - b_{12}x_2(t)], \\ \dot{x}_2(t) = x_2(t)[-a_2 + b_{21}x_1(t) - b_{22}x_2(t) - b_{23}x_3(t)], \\ \dot{x}_3(t) = x_3(t)[-a_3 + b_{32}x_2(t) - b_{33}x_3(t)], \end{cases} \tag{1}$$

where $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the densities of prey, predator and top-predator population at time t respectively. The parameters a_1, a_2 and a_3 are positive constants that stand for the intrinsic growth rate of the species $x_1(t)$, the death rate of the species $x_2(t)$, and the death rate of the species $x_3(t)$, respectively. The coefficients b_{11}, b_{22}, b_{33} are the intra-specific competition in the resource, b_{12}, b_{23} represent the rate of consumption and b_{21}, b_{32} represent the contribution of prey to the growth of predator.

Actually, the growth of populations in the natural world is always affected by environmental stochastic perturbations which should be taken into consideration in the process of mathematical modelling. In the literatures, many authors have studied population systems affected by white noise [7–16]. Especially, Takeuchi [9] investigated the evolution of a system composed of two predator–prey deterministic systems described by Lotka–Volterra equations in random environment. Ji [11] considered a predator–prey model with modified Leslie–Gower and Holling-type II schemes with stochastic perturbation. Nguyen [16] investigated a stochastic ratio-dependent predator–prey model. Inspired by the above literatures, in this paper, we consider the effect of white noise on the three-species food chain. From model (1) one can derive the following model with stochastic perturbations:

$$\begin{cases} dx_1(t) = x_1(t)\big[\big(a_1 - b_{11}x_1(t) - b_{12}x_2(t)\big)dt + \sigma_1 dB(t)\big], \\ dx_2(t) = x_2(t)\big[\big(-a_2 + b_{21}x_1(t) - b_{22}x_2(t) - b_{23}x_3(t)\big)dt + \sigma_2 dB(t)\big], \\ dx_3(t) = x_3(t)\big[\big(-a_3 + b_{32}x_2(t) - b_{33}x_3(t)\big)dt + \sigma_3 dB(t)\big], \end{cases} \tag{2}$$

where $B(t)$ are white noises, and σ_i is a positive constant representing the intensity of the white noise. We always assume that σ_i is not all equal. The existence, uniqueness and non-extinction property of the solution of system (2) have been studied in [15]. We replace model (2) by a slightly simpler one. Let $u_1 = \ln x_1, u_2 = \ln x_2, u_3 = \ln x_3$. Then, by Itô's formula, the random variables u_1, u_2, u_3 satisfy

$$\begin{cases} du_1(t) = \big(c_1 - b_{11}e^{u_1} - b_{12}e^{u_2}\big)dt + \sigma_1 dB(t), \\ du_2(t) = \big(-c_2 + b_{21}e^{u_1} - b_{22}e^{u_2} - b_{23}e^{u_3}\big)dt + \sigma_2 dB(t), \\ du_3(t) = \big(-c_3 + b_{32}e^{u_2} - b_{33}e^{u_3}\big)dt + \sigma_3 dB(t) \end{cases} \tag{3}$$

where $c_1 = a_1 - \sigma_1^2/2$, $c_2 = a_2 + \sigma_2^2/2$, $c_3 = a_3 + \sigma_3^2/2$.

The aim of this paper is to study the long-time behaviour of the solutions. The long-time behaviour of system (3) depends on the constants $b_{11}, b_{12}, b_{21}, b_{22}, b_{23}, b_{32}, b_{33}, c_1, c_2, c_3$. The study reveals that the other dynamic scenarios of system (3) are characterized by those parameters. The main results are listed as follows:

- Under some conditions (see Theorem 3.1), we show that the density of the distribution of system (3) converge to a stationary density;
- If $c_1 < 0$, then $\lim_{t\to\infty} u_i(t) = -\infty,\ a.e.\ \ i = 1, 2, 3$ (see Theorem 3.10);
- If $c_1 > 0, b_{11}c_2 > b_{21}c_1$, then $\lim_{t\to\infty} u_i(t) = -\infty,\ a.e.\ \ i = 2, 3$, and the distribution of the process $u_1(t)$ converges weakly to the measure which has the density $f_*(x) = Cexp\{2c_1x/\sigma_1^2 - (2b_{11}/\sigma_1^2)e^x\}$ (see Theorem 3.9);
- If $c_1 > 0, b_{11}c_2 < b_{21}c_1, b_{11}b_{22}c_3 + b_{11}b_{32}c_2 > b_{21}b_{32}c_1$, then $\lim_{t\to\infty} u_3(t) = -\infty,\ a.e$, and there exists a unique density $\overline{U}_*(x, y)$ which is a stationary solution of the first two equations of system (3) (see Theorem 3.8).

In this paper we focus on the existence of stationary distribution of system (3). Since the Fokker–Planck equation corresponding to system (3) is of degenerate type, the approach used in [17] to obtain the existence of stationary distribution is invalid for system (3). Our new approach comes from Markov semigroup theory which was used in [18–21].

The rest of the paper is organized as follows: In Section 2, we present some auxiliary results concerning Markov semigroups. In Section 3, we formulate the main results and its proof. Finally, we illustrate some results through an example in Section 4.

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