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Original article

# The uniqueness theorem for cohomologies on the category of polyhedral pairs

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**Abstract**

Let $X$ be a topological space and $F = \{F_\alpha\}$ be a direct system of all compact subsets $F_\alpha$ of $X$, directed by inclusions. For any homology theory $H_*$ the groups $\{H_*(F_\alpha) \mid F_\alpha \subset X\}$ constitute a direct system, and the maps $H_*(F_\alpha) \to H_*(X)$ define a homomorphism $i_* : \varinjlim H_*(F_\alpha) \to H_*(X)$.

As is known (Theorem 4.4.6, Spanier, 1966), for the singular homology, the homomorphism $i_*$ is an isomorphism

$$i_* : \varinjlim H_*^s(F_\alpha) \xrightarrow{\sim} H_*^s(X). \tag{1}$$

Using the isomorphism (1), it is proved that for the homologies having compact support $H$ there is the uniqueness theorem on the category of polyhedral pairs (Theorem 4.8.14, Spanier, 1966).

Since the singular homology theory is a homology theory with compact supports, the uniqueness theorem connects all homology theories having compact supports with the singular homology theory.

Let $H^*$ be a cohomology theory. The groups $\{H^*(F_\alpha) \mid F_\alpha \subset X\}$ constitute an inverse system, and the maps $H^*(X) \to H^*(F_\alpha)$ define a homomorphism

$$i^* : H^*(X) \to \varprojlim H^*(F_\alpha).$$

Since the homology functor does not commute with inverse limits, it is not true that the singular cohomology of a space is isomorphic to the inverse limit of the singular cohomology of its compact subsets (that is, there is no general cohomology analogue of Theorem 4.4.6, Spanier, 1966).

In the present work, it will be shown that there is such connection for a singular cohomology. Namely, there exists a finite exact sequence

$$\begin{aligned} 0 &\longrightarrow \varprojlim{}^{(2n-3)} H_s^1(F_\alpha, G) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(1)} H_s^{n-1}(F_\alpha, G) \longrightarrow H_s^n(X, G) \\ &\longrightarrow \varprojlim H_s^n(F_\alpha, G) \longrightarrow \varprojlim{}^{(2)} H_s^{n-1}(F_\alpha, G) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(2n-2)} H_s^1(F_\alpha, G) \longrightarrow 0. \end{aligned} \tag{2}$$

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The terms the Alexander cohomology with compact supports and the singular cohomology with compact supports used in the works (Spanier, 1966; Mdzinarishvili, 1984) do not refer to our problem. Therefore, cohomology theory, in particular the singular cohomology, for which there is a finite exact sequence (2), is called a cohomology with partially compact supports.

In the present work, using a finite exact sequence (2), it is proved the uniqueness theorem for a cohomology having partially compact supports on the category of polyhedral pairs. Hence, the uniqueness theorem connects all cohomology theories with partially compact supports with the singular cohomology theory.



*Keywords:* Singular cohomology; A finite exact sequence; The category of polyhedral pairs

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Let $C_* = \{C_n\}$ be a chain complex of abelian groups $C_n$,

$$C_* = C_0 \overset{\partial_1}{\longleftarrow} C_1 \overset{\partial_2}{\longleftarrow} \cdots \longleftarrow C_{n-1} \overset{\partial_n}{\longleftarrow} C_n \overset{\partial_{n+1}}{\longleftarrow} \cdots . \tag{3}$$

We denote $Z_n = \operatorname{Ker} \partial_n$, $B_n = \operatorname{Im} \partial_{n+1}$, $H_n = Z_n / B_n = H_n(C_*)$.

Let $\operatorname{Hom}(-, G)$ be the contravariant functor, where $G$ is an abelian group. Using the chain complex $C_*$ from (3) and the functor $\operatorname{Hom}(-, G)$, we have a cochain complex $C^* = \operatorname{Hom}(C_*, G)$, where $C^n = \operatorname{Hom}(C_n, G)$ and $\delta^n : C^{n-1} \to C^n$. Denote also $Z^n = \operatorname{Ker} \delta^{n+1}$, $B^n = \operatorname{Im} \delta^n$, $H^n = Z^n / B^n = H^n(C^*)$.

**Lemma 1.** *If $C_*$ is a free chain complex, then there is an exact sequence*

$$0 \longrightarrow \operatorname{Hom}(B_{n-1}, G) \longrightarrow Z^n \longrightarrow \operatorname{Hom}(H_n, G) \longrightarrow 0. \tag{4}$$

**Proof.** Since $C_*$ is a free chain complex, $Z_n$ and $B_n$ are free abelian groups for $n \in \mathbb{Z}$. Consider the exact sequences

$$0 \longrightarrow Z_n \overset{i_n}{\longrightarrow} C_n \overset{j_n}{\longrightarrow} B_{n-1} \longrightarrow 0$$

and

$$0 \longrightarrow B_n \longrightarrow Z_n \overset{t_m}{\longrightarrow} H_n \longrightarrow 0.$$

Using the above sequences, the functor $\operatorname{Hom}(-, G)$, and also Theorems 3.3.2, 3.3.5 and Lemma 1.5.4 [1], we have, respectively, the exact sequences

$$0 \longrightarrow \operatorname{Hom}(B_{n-1}, G) \longrightarrow \operatorname{Hom}(C_n, G) \longrightarrow \operatorname{Hom}(Z_n, G) \longrightarrow 0, \tag{5}$$

$$0 \longrightarrow \operatorname{Hom}(H_n, G) \longrightarrow \operatorname{Hom}(Z_n, G) \longrightarrow \operatorname{Hom}(B_n, G) \longrightarrow \operatorname{Ext}(H_n, G) \longrightarrow 0. \tag{6}$$

The commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_n & \overset{k_n}{\longrightarrow} & Z_n & \overset{i_n}{\longrightarrow} & C_n \\ & & & {\scriptstyle j_{n+1}}\nwarrow & & \nearrow{\scriptstyle \partial_{n+1}} & \\ & & & & C_{n+1} & & \end{array} \tag{7}$$