Available online at www.sciencedirect.com
ScienceDirect

Transactions of A. Razmadze Mathematical Institute ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

On some methods of extending invariant and quasi-invariant measures

A. Kirtadze[a,b,*], N. Rusiashvili[b]

[a] *A. Razmadze Mathematical Institute I. Javakhishvili Tbilisi State University, 6 Tamarashvili st., Tbilisi 0177, GA, United States*
[b] *Department of Mathematics Georgian Technical University, 77 Kostava st., Tbilisi 0175, GA, United States*

Received 21 June 2017; accepted 1 August 2017
Available online xxxxx

Abstract

In the present paper an approach to some questions in the theory of invariant (quasi-invariant) measures is discussed. It is useful in certain situations, where given topological groups or topological vector spaces are equipped with various nonzero σ-finite left invariant (left quasi-invariant) measures.

Keywords: Invariant measure; Quasi-invariant measure; Extensions of measures; Surjective homomorphism

The measure extension problem is one of the most important questions in measure theory. It forms a basis for harmonic analysis, the theory of functions of a real variable, probability theory, the theory of dynamical systems, and many other domains of contemporary mathematics. An interesting and important direction in measure theory is concerned with the investigation of properties of various (countably-additive) extensions of initial measures.

In this connection, there are some well-known methods of extending invariant measures: Marczewski's method; the method of Kodaira and Kakutani; the method of Kakutani and Oxtoby; the method of surjective homomorphisms.

Various aspects of the theory of extensions of invariant (and, more generally, quasi-invariant) measures are widely presented in the works of many authors (see, [1–11]).

A measure μ defined on some G-invariant σ-algebra of subsets of $(G, \cdot)$ is called quasi-invariant with respect to G (briefly, G-quasi-invariant) if, for every μ-measurable set X and for each $g \in G$, the relation

$$\mu(X) = 0 \Leftrightarrow \mu(g \cdot X) = 0$$

* Corresponding author at: A. Razmadze Mathematical Institute I. Javakhishvili Tbilisi State University, 6 Tamarashvili st., Tbilisi 0177, GA, United States.

E-mail addresses: kirtadze2@yahoo.com (A. Kirtadze), nino.rusiashvili@gmail.com (N. Rusiashvili).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.08.002

holds true. Moreover, if the equality

$$\mu(g \cdot X) = \mu(X)$$

is valid for any μ-measurable X and for any $g \in G$, then μ is called an invariant measure with respect to G (briefly, G-invariant measure).

Above-mentioned problem has the following three aspects:

(1) purely set-theoretical aspect;

(2) algebraic aspect;

(3) topological aspect.

A sufficiently general method of extending measures was suggested by Marczewski. This method is purely set-theoretical because no specific properties of given measurable space are used. According to a result of Marczewski, we can always extend Lebesgue measure to an isometrically-invariant countably additive measure (see, for example [1]).

A. Kharazishvili applied a purely algebraic method of surjective homomorphisms and solved the analogue of W.Sierpinski's problem for nonzero sigma-finite quasi-invariant (invariant) measures on arbitrary uncountable solvable groups (see, [4]).

An important special case of the method of surjective homomorphisms is the method of direct products which can be described as follows.

Suppose that two groups $(G, \cdot)$ and $(H, \cdot)$ are given and a set $X \subset G$ has a "nice" measure-theoretical property with respect to G. Then, in some situations, it turns out that the set $X \times H$ preserves this property with respect to the direct product $G \times H$. Notice that here $(G, \cdot)$ and $(H, \cdot)$ are arbitrary groups (not necessarily commutative).

Example 1. The method of direct products is essential for studying the property of metrical transitivity (ergodicity) of given measure. In particular, if an invariant measure μ_1 is metrically transitive with respect to a countable transformation group G_1 and an invariant measure μ_2 is metrically transitive with respect to a transformation group G_2, then the product measure $\mu_1 \times \mu_2$ is metrically transitive with respect to the product group $G_1 \times G_2$. Since the metrical transitivity of a measure is closely connected with the uniqueness property, one can conclude that the method of direct products turns out to be helpful for establishing the uniqueness property of a given invariant measure.

About Example 1 see, [12,13].

Example 2. The method of direct products is useful for obtaining some generalizations of W. Sierpinski's old result for an uncountable group $(G, \cdot)$, with the regular $card(G) = \alpha$. In particular, let $(G, \cdot)$ be an arbitrary group such that

$$G = G_1 \cdot G_2 \quad (G_1 \cap G_2 = \{e\})$$

where G_1 and G_2 are subgroups of G and $card(G_1) = \omega_1$ and e denotes the neutral element of G. If μ is a nonzero σ-finite G-quasi-invariant measure on G, then for each uncountable set $X \subset G_1$, there exist a G-quasi-invariant measure μ' on G extending μ and a set $Y \in I(\mu')$, for which we have

$$X \cdot Y = G \notin I(\mu'),$$

where $I(\mu')$ is the σ-ideal generated by all μ'-measure zero sets in G.

In particular, if $X \in I(\mu')$, then G is representable in the form of algebraic product of two μ'-measure zero sets.

About Example 2 see, [14].

Example 3. The method of direct products is also useful for constructing non-separable extensions of invariant measures given on infinite-dimensional topological groups or topological vector spaces. In the infinite-dimensional topological vector space $\mathbf{R}^{\mathbf{N}}$, a nonzero σ-finite invariant Borel measure χ was constructed, which is metrically transitive with respect to a dense vector subspace of $\mathbf{R}^{\mathbf{N}}$. On the other hand, in the Euclidean space $\mathbf{R}^n$ there exists a non-separable metrically transitive invariant measure μ extending the standard Lebesgue measure λ_n in $\mathbf{R}^n$. By applying the method of direct products it can be shown that the product measure $\chi \times \mu$ is non-separable, invariant with respect to a dense vector subspace of $\mathbf{R}^{\mathbf{N}}$, and metrically transitive with respect to the same subspace. Consequently, the completion of $\chi \times \mu$ has the uniqueness property.

About Example 3 see, [5–7,15].

Please cite this article in press as: A. Kirtadze, N. Rusiashvili, On some methods of extending invariant and quasi-invariant measures, Transactions of A. Razmadze Mathematical Institute (2017), http://dx.doi.org/10.1016/j.trmi.2017.08.002.