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Original article

On the Wiener–Hopf factorization of rational matrices

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Abstract

The Wiener–Hopf factorization theorem for rational matrices is proved with respect to very general contours using purely algebraic method.

Keywords: Rational functions; Matrices; Modules; Vector bundles; Cohomologies

1. Introduction

The Wiener–Hopf factorization of a rational matrix G relative to a contour Γ refers to a decomposition

$$G = G^+ D G^-,$$

where the factor G^+ is analytic and regular on the inner domain Ω^+ of Γ, the factor G^- is analytic and regular on the outer domain Ω^-, and D is a diagonal matrix of the form

$$D = \mathrm{diag}(s^{n_1}, \ldots, s^{n_r}).$$

(Here $n_1, \ldots, n_r$ are integers, which are uniquely determined up to permutation.)

This kind of factorization was initiated by Wiener and Hopf in their famous paper [1], and it is a powerful tool by which one solves singular integral equations and related boundary value problems. There is a vast literature on Wiener–Hopf's factorization and its applications (see [2–4] and the references therein).

The factorization theorem by itself is purely algebraic in nature, and it is the goal of this article to prove it by a purely algebraic method. The goal is also to extend the theorem to a very general situation.

The situation considered is the following: The "inner domain" Ω^+ and the "outer domain" Ω^- are taken to be *arbitrary* nonempty subsets of the complex projective line covering it, and the "contour" Γ is defined to be $\Omega^+ \cap \Omega^-$.

The development, in fact, will be carried out for the case when the ground field is an arbitrary field $\mathbb{F}$, not necessarily the complex number field.

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2. Algebraic preliminaries

Let ξ_0 and ξ_1 be two indeterminates, and let $\mathbb{P}^1$ be the projective line over $\mathbb{F}$. The points in $\mathbb{P}^1$ are assumed to be the equivalence classes of irreducible homogeneous polynomials in $\mathbb{F}[\xi_0, \xi_1]$. (We remind that two such polynomials are equivalent if they differ by a nonzero constant factor.) If p is an irreducible homogeneous polynomial, then the corresponding point is denoted by $[p]$. The degree of a point is the degree of a defining polynomial. One defines the zero point and the infinite point respectively as

$$0 = [\xi_1] \quad \text{and} \quad \infty = [\xi_0].$$

The set of finite points, i.e., the set $\mathbb{P}^1 \setminus \{\infty\}$, is called the affine line and is denoted by $\mathbb{A}^1$.

Remark. Points of degree 1 are of special interest. They are determined by 1-forms $a_0\xi_1 - a_1\xi_0$ with $a_0, a_1 \in \mathbb{F}$ such that $(a_0, a_1) \neq (0, 0)$, and consequently may be identified with equivalence classes in $\mathbb{F}^2 \setminus \{(0, 0)\}$ (i.e., with points of the "classical" projective line).

A rational function is a ratio f/g, where f and $g \neq 0$ are homogeneous polynomials of the same degree. Of special importance are $s = \xi_1/\xi_0$ and $t = \xi_0/\xi_1$. Rational functions form a field; we shall denote it by K. It is worth noting that

$$K = \mathbb{F}(s) \quad \text{and} \quad K = \mathbb{F}(t).$$

Letting $Spec(\mathbb{F}[s])$ (resp. $Spec(\mathbb{F}[t])$) denote the set of monic irreducible polynomials in the polynomial ring $\mathbb{F}[s]$ (resp. $\mathbb{F}[t]$), we have canonical bijections

$$\mathbb{P}^1 \setminus \{\infty\} \simeq Spec(\mathbb{F}[s]) \quad \text{and} \quad \mathbb{P}^1 \setminus \{0\} \simeq Spec(\mathbb{F}[t]).$$

Given a point x, a (nonzero) rational function $\varphi = f/g$ can be written in the form

$$\varphi = p^n \cdot f_0/g_0,$$

where p is a defining polynomial of x, f_0 and g_0 are homogeneous polynomials prime to p and n is an integer. The integer n is uniquely determined, and one sets

$$ord_x(\varphi) = n.$$

The function $ord_x : K \setminus \{0\} \to \mathbb{Z}$ is surjective and satisfies the following two conditions

- $ord_x(\varphi\psi) = ord_x(\varphi) + ord_x(\psi)$,
- $ord_x(\varphi + \psi) \geq \min\{ord_x(\varphi), ord_x(\psi)\}$.

(This type of functions are called discrete valuations (see Ch. 9 in [5]).)

A nonzero rational function φ is said to be regular at a point x if $ord_x(\varphi) \geq 0$. Rational functions regular at x form a ring, denoted by O_x. (The zero rational function is regarded to be regular at all points.)

Throughout this article, Ω^+ and Ω^- are two fixed nonempty subsets of $\mathbb{P}^1$ that cover the latter:

$$\mathbb{P}^1 = \Omega^+ \cup \Omega^-.$$

Choose two points a^+ and a^- that have degree 1 and such that

$$a^+ \in \Omega^+ \setminus \Omega^- \quad \text{and} \quad a^- \in \Omega^- \setminus \Omega^+.$$

(Existence of two such points is assumed.) Define the "contour" Γ by setting

$$\Gamma = \Omega^+ \cap \Omega^-.$$

Here are a few interesting examples for the case when $\mathbb{F} = \mathbb{C}$.

Examples. (1) $\Omega^+ = \mathbb{C}$, $\Omega^- = \{\infty\}$, $\Gamma = \emptyset$, $a^+ = 0$, $a^- = \infty$.

(2) $\Gamma = \mathbb{C} \setminus \{0\}$, $\Omega^+ = \Gamma \cup \{0\}$, $\Omega^- = \Gamma \cup \{\infty\}$, $a^+ = 0$, $a^- = \infty$.

(3) $\Omega^+ = \mathbb{C}^+ \cup \infty$, $\Omega^- = \mathbb{C}^- \cup \infty$, $\Gamma = \mathbb{R} \cup \infty$, $a^+ = i$, $a^- = -i$.

The choice of a^+ and a^- is of no particular importance. There certainly exists a linear transformation of $\mathbb{P}^1$ that maps a^+ and a^- to 0 and ∞, respectively. Without loss of generality, we therefore make the following assumption.

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