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Original article

Some differential properties of anisotropic grand Sobolev–Morrey spaces

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Abstract

In this paper an anisotropic grand Sobolev–Morrey spaces are introduced. With the help of integral representation we study differential and differential-difference properties of functions from these spaces.

Keywords: Anisotropic grand Sobolev–Morrey spaces; Integral representation; Embedding theorem; Hölder spaces

1. Introduction and preliminary notes

This work is devoted to investigations of embedding theorems for anisotropic grand Sobolev–Morrey spaces $W^l_{p),a,\varkappa}(G)$ $\left(G \subset \mathbb{R}^n,\ l \in N^n,\ p \in (1,\infty),\ a \in [0,1],\ \varkappa \in (0,\infty)^n\right)$. First we introduce a grand Lebesgue–Morrey spaces and anisotropic grand Sobolev–Morrey spaces. By using integral representation for functions defined on n-dimensional domains satisfying the flexible horn condition [1], an embedding theorems for the latter spaces are proved.

The grand Lebesgue spaces $L_{p)}(G)$ for a measurable set $G \subset \mathbb{R}^n$ of finite Lebesgue measure were introduced in the work of T. Iwaniec and C. Sbordone in [2].

During the last 25 year a vast amount of research about grand Lebesgue and grand Lebesgue–Morrey spaces (with different norms) has been done by many mathematicians (see, e.g., [3–13]).

We note that in this paper it is investigated the anisotropic grand Sobolev–Morrey spaces $W^l_{p),a,\varkappa}(G)$ giving possibility to increase "Hölder exponents" (see Theorem 2) more than in the case space of the Sobolev–Morrey space $W^l_{p,a,\varkappa}(G)$ studied in [14].

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Suppose that $G \subset \mathbb{R}^n$ is bounded domain, $t > 0$, and for all $x \in \mathbb{R}^n$,

$$G_{t^\varkappa}(x) = G \cap \left\{ y : \left| y_j - x_j \right| < \frac{1}{2} t^{\varkappa_j},\ j = 1, 2, \ldots, n \right\}.$$

Definition 1. Denote by $W^l_{p),a,\varkappa}(G)$ the space of locally summable functions f on G having the generalized derivatives $D_i^{l_i} f$ ($l_i > 0$ are integers, $i = 1, 2, \ldots, n$) with the finite norm

$$\|f\|_{W^l_{p),a,\varkappa}(G)} = \|f\|_{p),a,\varkappa;G} + \sum_{i=1}^n \left\| D_i^{l_i} f \right\|_{p),a,\varkappa;G}, \tag{1}$$

where

$$\|f\|_{p),a,\varkappa;G} = \|f\|_{L_{p),a,\varkappa}(G)} = \sup_{\substack{0<t\leq d \\ x\in G \\ 0<\varepsilon<p-1}} \left(\frac{1}{t^{|\varkappa|a}} \frac{\varepsilon}{|G_{t^\varkappa}(x)|} \int_{G_{t^\varkappa}(x)} |f(y)|^{p-\varepsilon} dy \right)^{\frac{1}{p-\varepsilon}}, \tag{2}$$

$d = diam\ G$.

Definition 2 ([1]). For $\lambda \in (0, \infty)^n$, a domain $G \subset \mathbb{R}^n$ is called a domain with flexible λ-horn if for some $\delta \in (0, 1]$ and $T \in (0, \infty)$ for any $x \in G$ there exists a path

$$\rho\left(t^\lambda\right) = \rho\left(t^\lambda, x\right) = \left(\rho_1\left(t^{\lambda_1}, x\right), \ldots, \rho_n\left(t^{\lambda_n}, x\right)\right) \quad 0 \leq t \leq T,$$

with the following properties:

(a) For all $j \in \{1, 2, \ldots, n\}$, $\rho_j(u, x)$ is absolutely continuous on $\left[0, T^{\lambda_j}\right]$ and $\left|\frac{\partial \rho_j(u,x)}{\partial u}\right| \leq 1$ for all $u \in \left[0, T^{\lambda_j}\right]$;

(b) $\rho(0, x) = 0$ and $x + V(\lambda, x, \delta) = x + \bigcup_{0\leq t\leq T} \left[\rho\left(t^\lambda, x\right) + t^\lambda \delta^\lambda I\right] \subset G$, where $I = [-1, 1]^n$.

The set $x + V(\lambda, x, \delta)$ will be called a flexible λ-horn with vertex at the point x.

Observe some properties of $L_{p),a,\varkappa}(G)$ and $W^l_{p),a,\varkappa}(G)$.

1. The following embeddings hold:

$$L_{p),a,\varkappa}(G) \hookrightarrow L_{p)}(G), \qquad W^l_{p),a,\varkappa}(G) \hookrightarrow W^l_{p)}(G),$$

in particular,

$$\|f\|_{p),G} \leq \|f\|_{p),a,\varkappa;G}; \quad \|f\|_{W^l_{p)}(G)} \leq \|f\|_{W^l_{p),a,\varkappa}(G)}, \tag{3}$$

where

$$\|f\|_{W^l_{p)}(G)} = \|f\|_{p),G} + \sum_{i=1}^n \left\| D_i^{l_i} f \right\|_{p),G},$$

$$\|f\|_{p),G} = \|f\|_{L_{p)}(G)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|G|} \int_G |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}.$$

Indeed,

$$\|f\|_{p),a,\varkappa;G} = \sup_{\substack{0<t\leq d, \\ x\in G, \\ 0<\varepsilon<p-1}} \left(\frac{1}{[t]_1^{|\varkappa|a}} \frac{\varepsilon}{|G_{t^\varkappa}(x)|} \int_{G_{t^\varkappa}(x)} |f(y)|^{p-\varepsilon} dy \right)^{\frac{1}{p-\varepsilon}}$$

$$\geq d^{-\frac{|\varkappa|a}{p}} \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|G|} \int_G |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} = d^{-\frac{|\varkappa|a}{p}} \|f\|_{p),G}.$$

Note that the definition of the spaces $L_{p),a,\varkappa}(G)$ is equivalent to the expression

$$\|f\|_{p),a,\varkappa;G} = \sup_{\substack{0<t\leq 1 \\ x\in G \\ 0<\varepsilon<p-1}} \left(t^{-\frac{|\varkappa|a}{p-\varepsilon}} \|f\|_{p),G_{t^\varkappa}(x)} \right).$$