ARTICLE IN PRESS

Available online at www.sciencedirect.com
ScienceDirect

Transactions of A. Razmadze Mathematical Institute ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

The existence of solution for equilibrium problems in Hadamard manifolds

Salahuddin

Department of Mathematics, Jazan University, Jazan, Saudi Arabia

Received 19 December 2016; received in revised form 10 February 2017; accepted 19 April 2017
Available online xxxxx

Abstract

In this work, we consider iterative methods for solving a class of *equilibrium problems in Hadamard Manifolds* by using the auxiliary principle techniques. We also discuss the convergence of sequences generated by the algorithms.

Keywords: Equilibrium problems; Variational inequalities; Iterative methods; Algorithms; Existence theory; Hadamard manifolds

1. Introduction

Riemannian manifolds constitute a broad and fruitful framework for the development of different fields. Actually in the last decades concepts and techniques which fit in Euclidean spaces have been extended to this nonlinear framework. Most of the extended methods however require the Riemannian manifold to have non-positive sectional curvature. This is an important property which enjoyed by a large class of Riemannian manifolds and it is strong enough to imply light topological restriction and rigidity phenomena [1–3]. Particularly, Hadamard manifolds which are complete simply connected and finite dimensional Riemannian manifolds of non-positive sectional curvature, have been turned out to be a suitable setting for diverse disciplines. Hadamard manifolds are examples of hyperbolic spaces and geodesic spaces more precisely, a Busemann nonpositive curvature space and a $CAT(0)$ spaces, *see* [4–9].

Equilibrium problem theory provides us with a unified, natural, novel and general framework to study a wide class of problems, which arises in finance, economics, network analysis, transportation and optimization. This theory had applications across all disciplines of pure and applied sciences. Equilibrium problems include variational inequalities and related problems, *see* [10–14]. Very recently, much attention has been given to study the variational inequalities, variational inclusions, complementarity problems, equilibrium problems and related optimization problems on the Riemannian manifold and Hadamard manifold. Several idea and method from the Euclidean space have been extended and generalized to this nonlinear system. Hadamard manifolds are examples of hyperbolic spaces and geodesics, *see* [3–6,9,15–18].

E-mail address: salahuddin12@mailcity.com.
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.04.001
2346-8092/

In this paper, we used the auxiliary principle techniques to suggest and analyze an iterative method for solving the *equilibrium problems* on Hadamard manifolds. We also discuss the convergence of sequences generated by the algorithms.

2. Preliminaries

Let M be a simply connected m-dimensional manifold. Given $x \in M$, the tangent space of M at x is denoted by T_xM and the tangent bundle of M by $TM = \bigcup_{x\in M} T_xM$ which is naturally a manifold. A vector field A on M is a mapping of M into TM which associates to each point $x \in M$, a vector $A(x) \in T_xM$. We always assume that M can be endowed with a Riemannian metric to become a Riemannian manifold. We denote by $\langle\cdot,\cdot\rangle$ the scalar product on T_xM with the associated norm $\|\cdot\|_x$, where the subscript x will be omitted. Given a piecewise smooth curve $\gamma : [a, b] \longrightarrow M$ joining x to y (that is, $\gamma(a) = x$ and $\gamma(b) = y$), by using the metric we can define the length of γ as $L(\gamma) = \int_a^b \|\gamma'(t)\| dt$. Then for any $x, y \in M$, the Riemannian distance $d(x, y)$ which includes the original topology on M is defined by minimizing this length over the set of all such curves joining x and y. Let $\triangle$ be the Levi-Civita connection with $(M, \langle\cdot,\cdot\rangle)$. Let γ be a piecewise smooth curve in M. A vector field A is said to be parallel along γ if $\triangle_{\gamma'} A = 0$. If γ' itself is parallel along γ, we say that γ is a geodesic and in this case $\|\gamma'\|$ is a constant when $\|\gamma'\| = 1$, γ is said to be normalized. A geodesic $\gamma_{x,y}$ joining x to y in M is said to be minimal if its length equal to $d(x, y)$. A Riemannian manifold is complete if for any $x \in M$, all geodesics emanating from x are defined for all $t \in R$. By the Hopf–Rinow Theorem, we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover (M, d) is a complete metric space and bounded closed subsets are compact.

Let M be complete, then exponential map $exp_x : T_xM \longrightarrow M$ at x is defined by $exp_x v = \gamma_v(1, x)$ for each $v \in T_xM$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at x with velocity v (*i.e.*, $\gamma(0) = x$ and $\gamma'(0) = v$). Then $exp_x tv = \gamma_v(t, x)$ for each real number t. A complete simply connected Riemannian manifold of non-positive sectional curvature is called a Hadamard manifold. Throughout this paper, we always assume that M is an m-dimensional Hadamard manifold. The geodesic triangle $\triangle(x_1, x_2, x_3)$ of a Riemannian manifold is a set consisting of three points x_1, x_2, x_3 and three minimal geodesic joining these points.

Lemma 2.1 ([16]). *Let $x \in M$. Then $exp_x : T_xM \longrightarrow M$ is a diffeomorphism and for any two points $x, y \in M$ there exists a unique normalized geodesic $\gamma_{x,y}$ joining x to y, which is minimal.*

Lemma 2.2 ([5]). *Let $\triangle(x_1, x_2, x_3)$ be a geodesic triangle. Denote, for each $i = 1, 2, 3 \pmod 3$, $\gamma_i : [0, \ell_i] \longrightarrow M$ as the geodesic joining x_i to x_{i+1} and set $\alpha_i = L(\gamma_i'(0), -\gamma_{i-1}'(\ell_{i-1}))$, the triangle between the vectors $\gamma_i'(0)$ and $-\gamma_{i-1}'(\ell_{i-1})$, and $\ell_i = L(\gamma_i)$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \le \pi, \tag{2.1}$$

$$\ell_i^2 + \ell_{i+1}^2 - 2\ell_i\ell_{i+1}\cos\alpha_{i+1} \le \ell_{i-1}^2. \tag{2.2}$$

In terms of the distance and the exponential map, the inequality (2.2) can be rewritten as

$$d^2(x_i, x_{i+1}) + d^2(x_{i+1}, x_{i+2}) - 2\langle \exp^{-1}_{x_{i+1}} x_i, \exp^{-1}_{x_{i+1}} x_{i+2}\rangle \le d^2(x_{i-1}, x_i), \tag{2.3}$$

since

$$\langle \exp^{-1}_{x_{i+1}} x_i, \exp^{-1}_{x_{i+1}} x_{i+2}\rangle = d(x_i, x_{i+1})d(x_{i+1}, x_{i+2})\cos\alpha_{i+1}.$$

Lemma 2.3 ([19]). *Let $\triangle(x, y, z)$ be a geodesic triangle in a Hadamard manifold M. Then there exist $x', y', z' \in \mathbb{R}^2$ such that*

$$d(x, y) = \|x' - y'\|, \quad d(y, z) = \|y' - z'\|, \quad d(z, x) = \|z' - x'\|.$$

The $\triangle(x', y', z')$ is called the comparison triangle of the geodesic triangle $\triangle(x, y, z)$ which is unique up to isometry of M.

From the law of cosine of inequality (2.3), we have the following inequality:

$$\langle \exp_x^{-1} y,\ \exp_x^{-1} z\rangle + \langle \exp_y^{-1} x,\ \exp_y^{-1} z\rangle \ge d^2(x, y),\ \forall x, y, z \in M. \tag{2.4}$$

Please cite this article in press as: Salahuddin, The existence of solution for equilibrium problems in Hadamard manifolds, Transactions of A. Razmadze Mathematical Institute (2017), http://dx.doi.org/10.1016/j.trmi.2017.04.001.