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Original article

Integral equations of the third kind for the case of piecewise monotone coefficients

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Abstract

We examine the third kind integral equations in Hölder class. The coefficients of the equations are piecewise strictly monotone functions having simple zeros. By singular integral equations theory, for solvability of considered equations, we give the necessary and sufficient conditions. Finding a solution is reduced to solving a regular integral equation of second kind.

Keywords: Piecewise; Eigenvalues; Holomorphic; Singular operator

1. Introduction

The linear integral equation

$$A(x)\varphi(x) + \int_a^b K(x, y)\varphi(y)dy = f(x), \quad x \in]a, b[\tag{1}$$

where $A(x)$ has at least one zero is commonly called an equation of the third kind. Such equations acquire more and more significance in applied problems of mathematical physics. In particular, in kinetic theory, in transport theory, etc. (see [1]) and investigations in this area are of great interest. After the early works of Hilbert and also Picard there appeared a lot papers on equations of the third kind (see e.g. [2–5]). In this paper we present a method for solving Eq. (1) when the coefficient $A(x) \in C^1([a, b])$ is a piecewise strictly monotone function having simple zeros in]a, b[. Moreover, we assume that $A'(x) \in \mathbf{H}$ on $[a, b]$ the kernel $K \in \mathbf{H}$, on $[a, b] \times [a, b]$ and a free term $f \in \mathbf{H}^*$ (Muskhelishvili's class) [6]. Therefore we look for solutions $\varphi \in \mathbf{H}^*$ of this class to be more appropriate in certain applications. Our investigation is based on the spectral expansion ideas by Fridrichs [7] and Hilbert–Schmidt approach for the second kind self-adjoint equations. Methods of the theory singular integral operators are the basic methods for

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investigating [6]. We have applied this theory often to the similar type problems [8–18]. This paper is structured as follows: First, in Section 2, using the initial equation, we introduce integral operators and their corresponding integral equations which depend on the auxiliary parameter. Some of their properties, which will play an important role in further considerations, are investigated. The singular operator, which is connected to the introduced equation, is defined and its properties are studied in Section 3. In Section 4 the problem of reduction of the singular integral operator is studied. In Section 5, the Hilbert–Schmidt type expansion theorem assertion of an arbitrary function from $\mathbf{H}^*$ can be represented through the singular operator and the eigenfunctions depending on the parameter operator. In Section 6, analogous Hilbert–Schmidt theorems are proved depending on the parameter integral equation and main result is given for the initial equation. Without loss of generality, we assume that, $b > a$ in Eq. (1). This paper is in some sense a continuation of [8].

2. Preliminaries

Let $K(x, y)$ satisfy Hölder conditions on $[a, b] \times [a, b]$. Further assume that the function $g(z, x)$ defined in $(\mathbf{C} \setminus [m_A, M_A]) \times [a, b]$ where $m_A = \min A(x), \quad M_A = \max A(x), x \in [a, b]$ is holomorphic with respect to z and belongs to H with respect to x: Moreover $g(z, x)$ has boundary values

$$g^+(\zeta, x) = \lim_{z \to \zeta} g(z, x), \quad Rez > 0$$

and

$$g^-(\zeta, x) = \lim_{z \to \zeta} g(z, x), \quad Rez < 0, \quad \zeta \in [m_A, M_A].$$

Denote by $\mathbf{\Omega}$ operator $\mathbf{\Omega}: \quad g(z, x) \to (\mathbf{\Omega} g)(z, x)$,

$$(\mathbf{\Omega}_z g(z, \cdot))(z, x) := g(z, x) + \int_a^b \frac{K(x, y)}{A(y) - z} g(z, y) dy, \qquad x \in [a, b] \tag{2}$$

where z is an arbitrary complex number, $A(x) \in \mathbf{C}^1([a, b])$ is the piecewise strictly monotone real-valued function having simple zeros in $]a, b[$ and $A'(x) \in \mathbf{H}$. This operator operating on any function $g(z, x)$ piecewise holomorphic with respect to z with the cut on $[m_A, M_A]$ and satisfying the Holder condition with respect to x, will define with the cut on $[m_A, M_A]$ a piecewise holomorphic function.

Let $\zeta = A(x)$ be the piecewise strictly monotone function, we are able to partition the interval $]a, b[$ into subintervals $]c_{i-1}, c_i[, \quad i = \overline{1, n}, \quad c_0 = a, \; c_n = b$, such that in these subintervals the function $\zeta = A(x)$ will be strictly monotone and moreover $A'(c_i) = 0, \; i = \overline{1, n-1}$. Let $A_i^{-1}(\zeta)$ be an inverse function of $A(x)$ in subinterval $]c_{i-1}, c_i[$ i.e. $A_i^{-1}(A(x)) = x$ for $x \in [c_{i-1}, c_i], \quad i = \overline{1, n}$ and $A_i^{-1}(\zeta) \in [c_{i-1}, c_i]$.

Now, recall some properties of the Cauchy type integrals. In order to find the boundary values $(\Omega g)^{\pm}(\zeta, x)$ we apply the formulas analogous to formulas for the Cauchy type integrals [19].

Let

$$\Psi(z) = \frac{1}{2\pi i} \int_L \frac{P(\tau, z)}{Q(\tau, z)} d\tau$$

when P and Q are analytic functions with respect to z for all $\tau \in L$

1. P satisfies the Hölder condition with respect to τ
2. Q is the differentiability with respect to τ and $Q'_\tau(\tau, z) \in \mathbf{H}$
3. In the points when $Q(\tau, z) = 0$ we have $Q'_\tau(\tau, z) \neq 0$ and $Q'_z(\tau, z) \neq 0$.

Let $\varsigma = \psi(\tau)$ be solution of the equation $Q(\tau, z) = 0$ and $\tau = \omega(\varsigma)$ is its inverse, then write formulas

$$\Psi^{\pm}(w) = \pm \frac{1}{2} \frac{P(t, w)}{Q'_\tau(t, w)} + \frac{1}{2\pi i} \int_L \frac{P(\tau, w)}{Q(\tau, w)} d\tau$$

when $t = \omega(w)$.