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Original article

Harmonic analysis and integral transforms associated with a class of a system of partial differential operators

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Abstract

In this work, we consider a generalized system of partial differential operators, we define the related Fourier transform and establish some harmonic analysis results. We also investigate a wide class of integral transforms of Riemann–Liouville type. In particular we give a good estimate of these integrals kernels, inversion formula and a Plancherel theorem for the dual.

Keywords: Fourier transform; Convolution product; Integral operators; Mehler representation; Dual operator; Inversion formula

1. Introduction

The operator R_α defined by

$$R_\alpha(f)(r,x) = \begin{cases} \dfrac{2\alpha}{\Pi}\displaystyle\int_0^1\int_{-\Pi/2}^{\Pi/2} f(r\cos\theta, x + r\sin\theta)\cos^{2\alpha}\theta(1-s^2)^{\alpha-1}d\theta\, ds & \text{for } \alpha > 0 \\ \dfrac{1}{\Pi}\displaystyle\int_{-\Pi/2}^{\Pi/2} f(r\cos\theta, x + r\sin\theta)d\theta & \text{for } \alpha = 0, \end{cases}$$

and its dual ${}^tR_\alpha$ are of interest in several applications for example in image processing of the so-called aperture radar (SAR), data. . . [1], or in the linearized inverse scattering problems in acoustics [2,3]. These operators have been

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extensively studied in [4–14]. They arise in connection with the system

$$\begin{cases} \Delta_1 = \dfrac{\partial}{\partial x}, \\ \Delta_2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial x^2}, \end{cases} \tag{1.1}$$

of partial differential operators [15,6,16–19].

The main aim of this article is to define and study a wide class of integral transforms which generalize the operators R_α and ${}^tR_\alpha$; $\alpha \geq 0$.

More precisely, we consider the singular partial differential operators Δ_1 and $\Delta_{2,A}$ such that

$$\begin{cases} \Delta_1 = \dfrac{\partial}{\partial x}, \\ \Delta_{2,A} = \dfrac{\partial^2}{\partial r^2} + \dfrac{A'(r)}{A(r)}\dfrac{\partial}{\partial r} + \rho^2 - \dfrac{\partial^2}{\partial x^2}, \end{cases} \tag{1.2}$$

where ρ is non negative real number and A is a non negative function satisfying some properties.

First, we define a generalized Fourier transform $\mathcal{F}_A$ and generalized shift operator $\mathcal{T}_{(r,x)}$; $\Big((r,x) \in [0,\infty[\times\mathbb{R}\Big)$ related with $\Delta_{2,A}$. We give some harmonic analysis results associated with $\mathcal{F}_A$ and $\mathcal{T}_{(r,x)}$. Second we establish an integral representation of the eigenfunction of the operator $\Delta_{2,A}$. This result and by using the same techniques as Fitouhi [20,21], we define and study a wide class of integral transforms R_A and ${}^tR_A$ related with $\Delta_{2,A}$. More precisely we establish for these operators the same results given by Helgason [9], Ludwig [12] and Solmon [14] for the classical Radon transform on $\mathbb{R}^2$. Also, we define and characterize some spaces of functions on which R_A and ${}^tR_A$ are isomorphism.

The paper is arranged as follows. In Section 2, we recall some basic properties and results about the singular second order differential operator $\pounds_A = \frac{\partial^2}{\partial r^2} + \frac{A'(r)}{A(r)}\frac{\partial}{\partial r} + \rho^2$. In Section 3, we define a generalized Fourier transform $\mathcal{F}_A$ associated with the system (1.2) and we establish some harmonic analyses (inversion Formula, Paley–Wiener theorem and Plancherel theorem for $\mathcal{F}_A$). Also, we define and study a generalized shift operator $\mathcal{T}_{(r,x)}$; $(r,x) \in [0,\infty[\times\mathbb{R}$ and a generalized convolution product associated with $\mathcal{T}_{(r,x)}$. Section 4 deals with the integral representation of the eigenfunction related with $\Delta_{2,A}$ and the operator R_A and its dual ${}^tR_A$. In Section 5 we give an inversion formula for R_A, ${}^tR_A$ and Plancherel theorem for ${}^tR_A$.

2. Preliminaries of Chébli–Trimèche hypergroups

In this section we briefly recall some results of harmonic analysis related with the following second order singular differential operator on the half line:

$$\pounds = \pounds_A = \frac{\partial^2}{\partial r^2} + \frac{A'(r)}{A(r)}\frac{\partial}{\partial r} + \rho^2, \tag{2.3}$$

where A is continuous on $[0,\infty[$, twice continuously differentiable on $]0,\infty[$ and satisfies the conditions:

(1) $A(0) = 0$ and $A(x) > 0$ for $x > 0$.

(2) A is increasing and unbounded.

(3) $\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + B(x)$ on a neighborhood of 0, where $\alpha > \frac{-1}{2}$ and B is an odd C^∞-function on $\mathbb{R}$.

(4) $\frac{A'(x)}{A(x)}$ is a decreasing C^∞-function on $]0,\infty[$ and $\lim_{+\infty} \frac{A'(x)}{A(x)} = 2\rho \geq 0$.

(5) There exists a constant $\delta > 0$, satisfying

$$\begin{cases} \dfrac{A'(r)}{A(r)} = 2\rho + F(r)\exp(-\delta r), & \text{for } \rho > 0. \\ \dfrac{A'(r)}{A(r)} = \dfrac{2\alpha+1}{r} + F(r)\exp(-\delta r), & \text{for } \rho = 0, \end{cases}$$