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Original article

A companion of Ostrowski type inequalities for mappings of bounded variation and some applications

Hüseyin Budak*, Mehmet Zeki Sarikaya

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

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Abstract

In this paper, we establish a companion of Ostrowski type inequalities for mappings of bounded variation and the quadrature formula is also provided.

Keywords: Bounded variation; Ostrowski type inequalities; Riemann–Stieltjes integral

1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty, \tag{1.1}$$

for all $x \in [a, b]$ [1]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality*.

* Corresponding author.
E-mail addresses: hsyn.budak@gmail.com (H. Budak), sarikayamz@gmail.com (M.Z. Sarikaya).
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Definition 1. Let $P : a = x_0 < x_1 < \cdots < x_n = b$ be any partition of $[a, b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$. Then $f(x)$ is said to be of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i)|$$

is bounded for all such partitions. Let f be of bounded variation on $[a, b]$, and $\sum(P)$ denotes the sum $\sum_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a, b]$. The number

$$\bigvee_a^b (f) := \sup\left\{\sum(P) : P \in P([a, b])\right\},$$

is called the total variation of f on $[a, b]$. Here $P([a, b])$ denote the family of partitions of $[a, b]$.

In [2], Dragomir proved the following Ostrowski type inequalities for functions of bounded variation:

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then*

$$\left|\int_a^b f(t)dt - (b-a) f(x)\right| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \tag{1.2}$$

holds for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

Dragomir gave the following trapezoid inequality in [3]:

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then we have the inequality*

$$\left|\frac{f(a) + f(b)}{2}(b-a) - \int_a^b f(t)dt\right| \le \frac{1}{2}(b-a) \bigvee_a^b (f). \tag{1.3}$$

The constant $\frac{1}{2}$ is the best possible.

We introduce the notation $I_n : a = x_0 < x_1 < \cdots < x_n = b$ for a division of the interval $[a, b]$ with $h_i := x_{i+1} - x_i$ and $\nu(h) = \max\{h_i : i = 0, 1, \ldots, n-1\}$. Then we have

$$\int_a^b f(t)dt = A_T(f, I_n) + R_T(f, I_n) \tag{1.4}$$

where

$$A_T(f, I_n) := \sum_{i=0}^{n} \frac{f(x_i) + f(x_{i+1})}{2} h_i \tag{1.5}$$

and the remainder term satisfies

$$|R_T(f, I_n)| \le \frac{1}{2}\nu(h) \bigvee_a^b (f). \tag{1.6}$$

In [4], Dragomir proved the following companion Ostrowski type inequalities related functions of bounded variation: