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Original article

# Fixed point results for generalized $(\psi, \phi)$-weak contractions with an application to system of non-linear integral equations

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**Abstract**

The aim of this paper is to prove fixed point results under $(\psi, \phi)$-weak contractive condition for continuous weak compatible mappings in ordered $b$-metric spaces. The results proved herein generalize, modify and unify some recent results of the existing literature. An application demonstrating the usability of our established results is also discussed besides furnishing an illustrative example.


*Keywords:* Coincidence point; $(\psi, \phi)$-weak contraction; Weak compatible mapping and ordered $b$-metric spaces

## 1. Introduction and preliminaries

Alber and Guerre-Delabriere [1] established a novel fixed point result for weak contraction in Hilbert spaces. Rhoudes [2] extended this result to metric spaces and also showed the generality of such results besides deducing Banach contraction principle. In [3], Zhang and Song replaced the idea of $\phi$-weak contraction with generalized $\phi$-weak contraction and obtained their fixed point results in complete metric spaces. Dutta and Choudhury [4] proved some fixed point results in complete metric spaces under $(\psi, \phi)$-weak contractive condition whereas Doric [5] extended some fixed point results of [4,3] to generalized $(\psi, \phi)$-weak contraction. Abbas and Doric [6] proved similar results on fixed point in complete metric spaces involving four mappings while Murthy et al. [7] obtained fixed point results in complete metric spaces under $(\psi, \phi)$-generalized weak contractive condition.

The origin of existence results on fixed points in partially ordered metric spaces is often traced back to Ran and Reurring [8]. Using generalized weak contraction, Radenovic and Kadelburg [9] established certain fixed point results in partially ordered metric spaces. Radenovic et al. [10] and Salimi et al. [11] proved fixed point results besides

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discussing possible applications under cyclic contraction and cyclic $\alpha$–$\psi$–$\phi$-contraction respectively. In [12] Aghajani and Arab also discussed coupled coincidence point results under generalized $(\psi, \phi, \theta)$-almost contractive condition in ordered $b$-metric spaces. Furthermore, Roshan et al. [13] proved results on coincidence point for almost generalized and generalized $(\psi, \phi)$-weak contractions in partially ordered $b$-metric spaces. Aghajani et al. [14] utilized generalized weak contraction to prove their results on fixed points involving four mappings in partially ordered $b$-metric spaces. Huang et al. [15] established coincidence point results in partially ordered $b$-metric spaces without using a special lemma as employed in [13].

The aim of this paper is to prove common fixed point results for pair of weak compatible mappings satisfying generalized $(\psi, \phi)$-weak contractive condition in partially ordered $b$-metric spaces.

Throughout this paper, $\mathbb{R}^+$ stands for the set of non-negative real numbers.

**Definition 1.1** (*[16]*). Let $P, Q : Y \to Y$ be a pair of mappings on the partial order set $Y$. The pair $(P, Q)$ is called

(a) weakly increasing if $Pu \preceq Q(Pu)$ and $Qu \preceq P(Qu)$, $\forall\, u \in Y$,

(b) partially weakly increasing if, $\forall\, u \in Y$ $Pu \preceq Q(Pu)$.

**Definition 1.2** (*[17]*). Let $P, Q, S : Y \to Y$ be three mappings on the partial order set $(Y, \preceq)$ such that $P(Y) \subseteq S(Y)$ and $Q(Y) \subseteq S(Y)$. The pair $(P, Q)$ is called

(i) weakly increasing with respect to $S \Leftrightarrow \forall\, u \in Y, Pu \preceq Qw, \forall\, w \in S^{-1}(Pu)$ and $Qu \preceq Pw$ for all $w \in S^{-1}(Qu)$,

(ii) partially weakly increasing with respect to $S \Leftrightarrow Pu \preceq Qw, \forall\, w \in S^{-1}(Pu)$.

**Definition 1.3** (*[18]*). Let $P, Q : Y \to Y$ be a pair of mappings on a metric space $(Y, d)$. The pair $(P, Q)$ is said to be compatible if and only if

$$\lim_{m\to\infty} d(PQu_m, QPu_m) = 0,$$

whenever $\{u_m\}$ is a sequence such that,

$$\lim_{m\to\infty} Pu_m = \lim_{m\to\infty} Qu_m = r \text{ with } r \in Y.$$

**Definition 1.4** (*[19]*). Let $P, Q : Y \to Y$ be a pair of mappings on metric space $(Y, d)$. The pair $(P, Q)$ is weakly compatible when the pair $(P, Q)$ commutes on the set of coincidence points ($i.e.,$ $PQu = QPu$ when $Pu = Qu$).

**Definition 1.5** (*[20]*). Let $d_1 : Y \times Y \to \mathbb{R}^+$ be a mapping, where $Y$ is non-empty set. Then $d_1$ is called a $b$-metric if and only if ($\forall u$, $w$ and $v \in Y$ and $s \geq 1$) the following conditions are fulfilled:

($b_1$) $d_1(w, u) = 0$ if and only if $w = u$;

($b_2$) $d_1(w, u) = d_1(u, w)$;

($b_3$) $d_1(w, v) \leq s\big(d_1(w, u) + d_1(u, v)\big)$.

The pair $(Y, d_1)$ is called a $b$-metric space, where $d_1$ is termed as $b$-metric defined on a partial order set $(Y, \preceq)$. Such a $b$-metric space is called a partially ordered $b$-metric space.

**Definition 1.6** (*[13]*). A sequence $\{u_m\}$ is called $b$-Cauchy in $(Y, d_1)$ if and only if

$$\lim_{m,n\to\infty} d_1(u_m, u_n) = 0.$$

**Definition 1.7** (*[13]*). A sequence $\{w_m\}$ is called $b$-convergent in a $b$-metric space $(Y, d_1)$ if and only if there is $w \in Y$ such that $\lim_{m\to\infty} d(w_m, w) = 0$ ($i.e.,$ $\lim_{m\to\infty} w_m = w$).

**Lemma 1.8** (*[13]*). *Suppose that the sequences $\{u_m\}$ and $\{v_m\}$ are $b$-convergent to $u_1$ and $v_1$ respectively in $b$-metric space $(Y, d_1)$ with $s \geq 1$. Then,*

$$\frac{1}{s^2} d_1(u, v) \leq \lim_{m\to\infty} \inf d_1(u_m, v_m) \leq \lim_{m\to\infty} \sup d_1(u_m, v_m) \leq s^2 d_1(u, v).$$

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