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Original article

Unique fixed point results on closed ball for dislocated quasi G-metric spaces

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Abstract

The aim of this paper is to introduce the new concept of ordered complete dislocated quasi G-metric space. The notion of dominated mappings is applied to approximate the unique solution of non linear functional equations. In this paper, we find the fixed point results for mappings satisfying the locally contractive conditions on a closed ball in an ordered complete dislocated quasi G-metric space. Our results improve several well known classical results.

Keywords: Fixed point; Banach mapping; Kannan mapping; Closed ball; Dominated mapping; Ordered complete dislocated quasi metric spaces

1. Introduction and preliminaries

Let $T : X \to X$ be a mapping. A point $x \in X$ is called a fixed point of T if $x = Tx$. Let x_0 be an arbitrary chosen point in X. Define a sequence $\{x_n\}$ in X by a simple iterative method given by $x_{n+1} = Tx_n$, where $n \in \{0, 1, 2, 3, \ldots\}$. Such a sequence is called a picard iterative sequence and its convergence plays a very important role in proving existence of a fixed point of a mapping T. A self mapping T on a metric space X is said to be a Banach contraction mapping if,

$$d(Tx, Ty) \leq kd(x, y)$$

holds for all $x, y \in X$ where $0 \leq k < 1$. Recently, many results appeared related to fixed point theorem in complete metric spaces endowed with a partial ordering in literature. Ran and Reurings [1] proved an analogue of Banach's fixed

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point theorem in metric space endowed with partial order and gave applications to matrix equations. Recently, Arshad et al. [2] proved a result concerning the existence of fixed points of a mapping satisfying a contractive conditions on closed ball in a complete dislocated metric space. For further results on closed ball we refer the reader to [3–7] and references therein. Subsequently, Nieto et al. [8] extended the results of [1] for non decreasing mappings and applied this results to obtain a unique solution for a 1st order ordinary differential equation with periodic boundary conditions. On the other hand in 2005, Mustafa and Sims in [9] introduce the notion of a generalized metric space as generalization of the usual metric space. Mustafa and others studied fixed point theorems for mappings satisfying different contractive conditions for further useful results can be seen in [10–15]. Recently, Agarwal and Karapinar introduced some coupled fixed point theorems in G metric space [16]. Azam and Nayyar proved fixed point theorems for multivalued mappings in G-cone metric space see [17]. Further latest fixed point results on G metric space can be seen in [18–20]. The dominated mapping [21] which satisfies the condition $fx \preceq x$ occurs very naturally in several practical problems. For example x denotes the total quantity of food produced over a certain period of time and $f(x)$ gives the quantity of food consumed over the same period in a certain town, then we must have $fx \preceq x$.

In this paper we have obtained fixed point theorems for a contractive dominated self-mapping in an ordered complete dislocated quasi G-metric space on a closed ball to generalize, extend and improve some classical fixed point results. We have used weaker contractive condition and weaker restrictions to obtain unique fixed point.

Definition 1. Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:
(i) If $G(x, y, z) = G(y, z, x) = G(z, x, y) = 0$, then $x = y = z$,
(ii) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).
Then the pair (X, G) is called the dislocated quasi G-metric space. It is clear that if
$G(x, y, z) = G(y, z, x) = G(z, x, y) = 0$ then from (i) $x = y = z$. But if $x = y = z$ then $G(x, y, z)$ may not be 0. It is observed that if $G(x, y, z) = G(y, z, x) = G(z, x, y)$ for all $x, y, z \in X$, then (X, G) becomes a dislocated G-metric space.

Example 2. If $X = R^+ \cup \{0\}$ then $G(x, y, z) = x + \max\{x, y, z\}$ defines a dislocated quasi metric on X.

Definition 3. Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G-convergent to x. Thus, if $x_n \to x$ in a dislocated quasi G-metric space (X, G), then for any $\in > 0$, there exists $n, m \in N$ such that $G(x, x_n, x_m) < \in$, for all $n, m \geq N$.

Definition 4. Let (X, G) be a dislocated quasi G-metric space. A sequence $\{x_n\}$ is called G-Cauchy sequence if, for each $\in > 0$ there exists a positive integer $n^\star \in N$ such that $G(x_n, x_{m,} x_l) < \in$ for all $n, l, m \geq n^\star$; i.e. if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 5. A dislocated quasi G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.

Proposition 6. *Let (X, G) be a dislocated quasi G-metric space, then the following are equivalent:*

(1) *$\{x_n\}$ is G convergent to x.*
(2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*
(3) *$G(x_n, x, x) \to 0$ as $n \to \infty$.*
(4) *$G(x_n, x_m, x) \to 0$ as m $n \to \infty$.*

Definition 7. Let (X, G) be a G-metric space then for $x_0 \in X$, $r > 0$, the G-ball with centre x_0 and radius r is,

$$\overline{B(x_0, r)} = \{y \in X : G(x_0, y, y) \leq r\}.$$

Definition 8 (*[21]*)**.** Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

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